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SUPERSTRINGS AND SUPERMEMBRANES IN THE DOUBLY SUPERSYMMETRIC GEOMETRICAL APPROACH

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Abstract

We perform a generalization of the geometrical approach to describing extended objects for studying the doubly supersymmetric twistor–like formulation of super–p–branes. Some basic features of embedding world supersurface into target superspace specified by a geometrodynamical condition are considered. It is shown that the main attributes of the geometrical approach, such as the second fundamental form and extrinsic torsion of the embedded surface, and the Codazzi, Gauss and Ricci equations, have their doubly supersymmetric counterparts. At the same time the embedding of supersurface into target superspace has its particular features. For instance, the embedding may cause more rigid restrictions on the geometrical properties of the supersurface. This is demonstrated with the examples of an N=1 twistor–like supermembrane in D=11 and type II superstrings in D=10, where the geometrodynamical condition causes the embedded supersurface to be minimal and puts the theories on the mass shell.

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Introduction

Finding the most adequate way to describe physical objects is an important problem which, very often, allows one to achieve deeper knowledge and perform further development of the corresponding theory. One of the most typical examples is the theory of strings and superstrings, various formulations of which throw light on different features of the string.

Among the string formulations there is a so called geometrical approach, which is essentially based on the theory of surfaces embedded into a target space. This approach was originated in papers by Lund and Regge [1] and Omnes [2], and revealed a connection of the string equations of motion with two–dimensional (exactly solvable) non–linear equations, such as the sin–Gordon and Liouville equation.

Though, of course, all string formulations imply that string world–sheet is a surface embedded into a target space–time, the geometrical approach explores this in the most direct way by dealing with such objects as a target–space moving frame at every point of the surface, extrinsic curvature and torsion of the surface, and reducing the string equations to the system of the Codazzi, Gauss and Ricci equations completely determining the embedding of the surface.

The geometrical approach was studied in connection with the problem of formulating consistent quantum string theory in non–critical space–time dimensions and has been developed in application to strings and p–branes in a number of papers (see [3, 4] and references therein).

The interest to the approach is due to the deep relationship of p-brane equations of motion with equations describing non–linear systems such as σ–models in $\frac{SO(1,D-1)}{SO(1,p)\times SO(D-p-1)}$ target space, and exactly solvable and completely integrable dynamical systems (in the case of strings) [1, 2, 3, 4]. In particular, it is remarkable that choosing a Lorentz–covariant gauge, one can reduce the number of the string coordinates in a D–dimensional space–time to $(D-2)$ independent variables subject to a system of non–linear differential equations [3, 4] for which the general solution can be constructed.

As to membranes, a relation between their non–linear equations of motion and that of integrable systems has been found as well [5]. And since the problem of complete solving the membrane equations of motion is still open, the attempts to reformulate membrane theory directly in the geometrical framework of surface theory seem to be justified. Using this approach one may hope to find new physically interesting solutions to the membrane equations, and gain deeper insight into the problem of string–membrane duality [12, 13, 14].

To develop the geometrical approach, in addition to p–brane space–time coordinates one introduces auxiliary world surface fields describing a target space moving frame attached to every point of the world surface, so that a system of equations specifying the parallel transport of the moving frame along the world surface is equivalent to the p–brane

equations of motion. This determines a geometry on the surface induced by embedding. Note that the intrinsic geometry of the world surface (and the corresponding part of the induced geometry) characterize internal properties of the surface and the local gauge symmetries of the model, while the extrinsic part of the induced geometry specifies the motion of the p–brane in the target space.

Moving frame components can be introduced directly into a p–brane action [1] the latter being considered as a dynamical ground for the geometrical approach [1, 2, 3].

Here the question arises what is the natural way for introducing the moving frame into the super–p–brane theory [6]–[11].

One of the possibilities is a twistor–like formulation of super–p–branes [18]–[38] which provided the geometrical solution [23] to the problem of local fermionic κ–symmetry [45, 46, 47].

The twistor–like formulation is based on a notion of double supersymmetry originally introduced for constructing more general supersymmetric models [48] studied, in particular, in connection with the problem of coupling worldsheet supergravity to target space supergravity for unique treatment of the Neveu–Schwarz–Ramond and Green–Schwarz superstrings [24, 25, 30].

In the doubly supersymmetric formulation of super–p–branes auxiliary commuting spinor variables, having properties of twistors [17]–[23] and Lorentz harmonics [60],[51]–[53, 39, 44, 49, 50, 40, 42] appear as superpartners of the target superspace Grassmann coordinates, their bilinear combinations forming Lorentz vectors which can be identified with components of local moving frame in the target superspace. This provides the ground for a generalization and a development of the geometrical approach, which implies studying the embedding of a world supersurface into a target superspace.

In the present paper we perform the first steps in this direction and consider as examples an $N = 1$ supermembrane in $D = 11$ and superstrings in $D = 10$.

In the doubly supersymmetric formulation of super–p–branes the embedding of a world supersurface into a target superspace is specified by a geometrodynamical condition (see section 2.2), which prescribes the pullback of a target superspace one–form onto the world supersurface to have zero components along Grassmann directions of the latter [23]–[38]. The twistor–like solution to the Virasoro constraints arises as an integrability condition for the geometrodynamical equation. In the case of the D=11, N=1 supermembrane and the D=10, N=2 superstring imposing the geometrodynamical condition puts the theory on the mass shell, which causes the problem with constructing worldsheet superfield actions, as was noticed by Galperin and Sokatchev [36].

Below, when considering the doubly supersymmetric p–branes we will not discuss the problem of getting the action, since for our purpose of developing the geometrical approach just the equations of motion of super–p–branes are required. So for the two

[1] P–brane models of this kind have been considered, for example, in [15, 16].

theories under consideration the geometrodynamical condition can be regarded as one determining a minimal supersurface in a target superspace, and we will use it as the starting point for getting geometrical equations analogous to the Codazzi, Gauss and Ricci equation.

Leaving apart the problem of constructing the superfield action we also will not touch one important ingredient of the super–p–branes in the Green–Schwarz [6, 47] as well as the twistor–like superfield [29]–[38] formulation, namely a Wess–Zumino term and a corresponding Wess–Zumino differential form. In the Green–Schwarz formulation the crucial role of the Wess–Zumino term in the action is to ensure the local fermionic κ–symmetry. In the twistor–like action, in addition to the geometrodynamical term, the pullback of the Wess-Zumino form further specifies the embedding of the world supersurface and generates super–p–brane tension, thus turning a null super–p–brane [33] into the valuable extended object [32]. The Wess–Zumino term is a differential form on the world supersurface which is a closed form on the mass shell provided the geometrodynamical condition takes place (see Tonin in [29], and [32]), and when one gets the equations of motion from a super–p–brane action they contain the contribution from the Wess–Zumino term. Thus, as soon as the equations of motion are obtained (for instance, as a consequence of the geometrodynamical condition) the Wess–Zumino term does not provide any new information.

The paper is organized as follows.

In Chapter 1 we review the main features of the geometrical approach to bosonic p–branes by introducing the notion of the local frame, presenting an appropriate p–brane action to start with and rewriting the p–brane equations of motion in the form of the Codazzi, Gauss and Ricci equations for the second fundamental form and extrinsic torsion of world surface embedded into target space–time.

In Chapter 2 we perform a generalization of the geometrical approach to the case of super–p–branes. It is shown that the basic role in the formulation is played by a spinor local frame in target superspace the local vector frame being composed of the spinor one. The embedding of world supersurface is specified by the geometrodynamical condition. The supersymmetric analogues of the Codazzi, Gauss and Ricci equations and of the second fundamental form are considered. A condition for the embedded supersurface to be minimal is found.

In Chapter 3 and 4 the results of Chapter 2 are applied for studying particular features of D=11 N=1 supermembranes and D=10 type II superstrings, and it is shown that world supersurface embedding specified by the geometrodynamical condition is minimal in contrast to the case of a heterotic string.

In Conclusion we sum up the results obtained.

Our notation and convention are as follows. The small Latin indices stand for vectors and the Greek indices stand for spinors. All underlined indices correspond to target

(super)space of D bosonic dimensions, and that which are not underlined correspond to world (super)surface of (p+1) bosonic dimensions. The indices from the beginning of the alphabets denote the vector and spinor components in the tangent (super)space. Indices from the second half of the alphabets are world indices:

$$\underline{a},\underline{b},\underline{c} = 0,...,D-1 \qquad \underline{l},\underline{m},\underline{n} = 0,...,D-1;$$

$$a,b,c = 0,...,p \qquad l,m,n = 0,...,p$$

$$\underline{\alpha},\underline{\beta},\underline{\gamma} = 1,...,2^{[\frac{D}{2}]} \ \ (or \ 2^{[\frac{D}{2}-1]}) \qquad \underline{\mu},\underline{\nu},\underline{\rho} = 1,...,2^{[\frac{D}{2}]} \ \ (or \ 2^{[\frac{D}{2}-1]});$$

$$\alpha,\beta,\gamma = 1,...,2^{[\frac{p}{2}]} \qquad \mu,\nu,\rho = 1,...,2^{[\frac{p}{2}]}$$

$i,j,k = 1,...,D-p-1$ stand for the vector representation of SO(D-p-1);
p,q,r $(or\ \dot{p},\dot{q},\dot{r}) = 1,...,D-p-1$ stand for a spinor representation of SO(D-p-1).

More information about notation and convention the reader may find in the main text or in the Appendices.

Chapter 1

Geometrical approach to bosonic p–branes

1.1 Moving frame on the embedded surface

To describe an embedding of a $(p+1)$–dimensional world surface into a D–dimensional flat space–time one introduces in the target space (parametrized by $x^{\underline{m}}$) a local moving frame $u^{\underline{a}}(x^{\underline{m}}) \equiv dx^{\underline{n}} u_{\underline{n}}^{\underline{a}}(x^{\underline{m}})$, whose components $u_{\underline{n}}^{\underline{a}}(x^{\underline{m}})$ satisfy the orthonormality conditions

$$u_{\underline{m}}^{\underline{a}} \eta^{\underline{mn}} u_{\underline{n}\underline{b}} = \eta^{\underline{ab}} = diag(1, -1, \ldots, -1) \qquad (1.1)$$

Eq. (1.1) restricts the matrix $||u_{\underline{n}}^{\underline{a}}(x^{\underline{m}})||$ to take its values in the Lorentz group $SO(1, D-1)$. Thus, in particular, using an appropriate Lorentz transformations in the tangent space, one can always choose a frame whose components form the unit matrix $\delta_{\underline{m}}^{\underline{a}}$ globally in the flat space.

On the other hand, by use of the Lorentz transformations one can adjust a local frame to the embedded surface $x^{\underline{m}} = x^{\underline{m}}(\xi^m)$ in such a way that $d = (p+1)$ vectors $u_{\underline{m}}^{\ a}$ of the frame are parallel and $(D-p-1)$ vectors $u_{\underline{m}}^{\ i}$ are orthogonal to the surface at each point of the latter:

$$\partial_m x^{\underline{m}} u_{\underline{m}}^{\ i} = 0 \qquad (1.2)$$

$$\partial_m x^{\underline{m}} u_{\underline{m}}^{\ a} \quad \sim \quad e_m^{\ a}, \qquad (1.3)$$

where $e^{\ a} \equiv d\xi^m e_m^{\ a}(\xi)$ is an intrinsic vielbein form on the surface (locally parametrized by ξ^m). Eqs. (1.2), (1.3) determine the moving frame $u_{\underline{m}}^{\underline{a}}$ up to a local transformations of the subgroup $SO(1,p) \times SO(D-p-1)$ of the Lorentz group, SO(1,p) being identified with the structure group of the surface. Thus, $u_{\underline{m}}^{\underline{a}}$ can be regarded as Lorentz harmonics parametrizing the coset space $\frac{SO(1,D-1)}{SO(1,p)\times SO(D-p-1)}$.

If we consider a $d = p+1$ – dimensional surface as one created by a p–brane moving in space–time, Eqs. (1.2), (1.3) should be regarded as p–brane equations of motion derived

from the action principle. An appropriate action is [40, 41, 42, 43]

$$S_{D,p} = (\alpha')^{-\frac{1}{2}} \int d^{p+1}\xi \quad e(\xi) \left(-e^m_a \partial_m x^{\underline{m}} u^a_{\underline{m}} + p \right), \tag{1.4}$$

or in the language of differential forms:

$$S_{D,p} = -\frac{(-1)^p (\alpha')^{-\frac{1}{2}}}{p!} \int u^{a_1} e^{a_2} ... e^{a_{(p+1)}} \varepsilon_{a_1 a_2 ... a_{(p+1)}} + \frac{(-1)^p (\alpha')^{-\frac{1}{2}} p}{(p+1)!} \int e^{a_1} e^{a_2} ... e^{a_{(p+1)}} \varepsilon_{a_1 a_2 ... a_{(p+1)}}. \tag{1.5}$$

Here α' is a dimensional constant (the Regge slope parameter for the string case $p = 1$). The wedge product and the exterior derivative of the differential forms are implied in eq.(1.5) and below where applicable.

Eq.(1.4) (or (1.5)) is classically equivalent to the conventional p–brane action [55, 56, 57, 58] (see below). One can see that the first term in (1.5) differs from the second (cosmological) term only by the replacement of one of the vielbein forms e^a with the one–form $u^a = dx^{\underline{m}} u^a_{\underline{m}}$, where $u^a_{\underline{m}}$ are the $d = p+1$ orthonormal vectors from the moving frame (1.1) which transform under $SO(1,p)$. Note that *a priori* u^a is independent of e^a , and eqs. (1.2), (1.3) arise as the equations of motion of $u^a_{\underline{m}}$. Since $u^a_{\underline{m}}$ are world–surface fields subject to (1.1), one must take into account (1.1) when varying (1.4) with respect to $u^a_{\underline{m}}$. This can be performed either by explicit including the constraint (1.1) into the action, or by taking only such variations of $u^a_{\underline{m}}$, which do not break (1.1). We shall use the latter procedure which turns out to be more convenient, especially when dealing with spinor Lorentz harmonics [59], [39]–[44].

Apparently, the variations of $u^{\underline{a}}_{\underline{m}}$ which do not violate (1.1) are determined by quantities $O^{\underline{ab}} = -O^{\underline{ba}}$ taking their values in the algebra of $SO(1, D-1)$:

$$\delta u^{\underline{a}}_{\underline{m}} = u_{\underline{mb}} O^{\underline{ba}}. \tag{1.6}$$

For the differentials of $u^{\underline{a}}_{\underline{m}}$ we get the same expressions as (1.6):

$$du_{\underline{m}}^{\ \underline{a}} = u_{\underline{mb}} \Omega^{\underline{ba}}(d), \tag{1.7}$$

where $\Omega^{\underline{ab}}(d) = u^{\underline{a}}_{\underline{m}} du^{\underline{b}}_{\underline{m}}$ are $SO(1, D-1)$ Cartan forms.

For the vectors parallel and orthogonal to the world surface eq.(1.7) splits as follows:

$$du_{\underline{m}}^{\ a} = u_{\underline{m}b} \Omega^{ba}(d) + u_{\underline{m}k} \Omega^{ka}(d) \tag{1.8}$$

$$du_{\underline{m}}^{\ i} = u_{\underline{m}b} \Omega^{bi}(d) + u_{\underline{m}k} \Omega^{ki}(d), \tag{1.9}$$

where the one–forms $\Omega^{ab}(d)$ and $\Omega^{ik}(d)$ take their values in the $SO(1,p)$ and $SO(D-p-1)$ subalgebra of the $SO(1, D-1)$ algebra, respectively, and $\Omega^{ai}(d)$ belong to the cotangent space of the coset space $\frac{SO(1,D-1)}{SO(1,p) \times SO(D-p-1)}$.

By definition $\Omega^{\underline{ab}}(d)$ satisfy the Maurer–Cartan equations:

$$d\Omega^{ai} - \Omega^a{}_b \Omega^{bi} + \Omega^{aj} \Omega^{ji} = 0, \tag{1.10}$$

$$R^{ab}(d,d) = d\Omega^{ab} - \Omega^{a}_{\ c}\Omega^{cb} = \Omega^{ai}\Omega^{bi}, \tag{1.11}$$

$$R^{ij}(d,d) = d\Omega^{ij} + \Omega^{ij'}\Omega^{j'j} = \Omega^{ai}\Omega_{a}^{\ j}, \tag{1.12}$$

where d is the right external differential ($d(\Omega_p\Omega_q) = \Omega_p d\Omega_q + (-1)^q d\Omega_p\Omega_q$ for a product of any p– and q– form).

Now we are ready to show the classical equivalence of the p–brane formulation considered and the conventional one [55]–[58] (see [41] for the string case).

The equations $\delta S/\delta e^{m}_{a} = 0$ and $\int \delta u_{\underline{m}}^{\ a}\delta S/\delta u_{\underline{m}}^{\ a} = 0$ (with $\delta u_{\underline{m}}^{\ a}$ from (1.6)) have the form

$$\partial_m x^{\underline{m}} u_{\underline{m}}^{\ a} = e_{m}^{\ a}, \tag{1.13}$$

$$\partial_m x^{\underline{m}} u_{\underline{m}}^{i} = 0, \tag{1.14}$$

in which one can recognize the embedding conditions (1.2), (1.3). By use of the orthonormality conditions (1.1) eqs. (1.13), (1.14) can be rewritten as follows

$$dx^{\underline{m}} = e_a u^{\underline{m}a}, \tag{1.15}$$

or

$$u^{\underline{m}a} = e^{am}\partial_m x^{\underline{m}}, \tag{1.16}$$

Finally, varying (1.4) with respect to $x^{\underline{m}}$ one gets

$$\partial_m \left(e e_{a}^{\ m} u^{\underline{m}a}\right) = 0. \tag{1.17}$$

Substituting eq. (1.16) back into (1.4) and introducing a world surface metric

$$g_{mn} \equiv e_{ma}e_{n}^{\ a} = \partial_m x^{\underline{m}}\partial_n x_{\underline{m}}$$

we derive a conventional action functional for bosonic p–branes [55] – [58]:

$$S_{conventional} = \int d\xi^{(p+1)} det^{1/2}(\partial_m x^{\underline{m}}\partial_n x_{\underline{m}}). \tag{1.18}$$

In conclusion to this section we would like to draw attention to the fact that the bosonic p-brane action in the form (1.4) does not possess Weyl invariance ($e^{a}_{m} \rightarrow We^{a}_{m}$) even for the case of strings (p=1) (in contrast to the string action $\int d\xi^2 e g^{mn}\partial_m x^{\underline{m}}\partial_n x_{\underline{m}}$ which also involves the intrinsic worldsheet metric). In this respect (1.4) is closer to the Nambu action (1.18). To get a Weyl invariant action we should introduce into (1.4) (or (1.5)) an auxiliary field $W(\xi)$ in such a way that the first and the second term acquire the factor W^p and W^{p+1}, respectively:

$$S_{D,p} = (\alpha')^{-\frac{1}{2}} \int d^{p+1}\xi \quad e(\xi)\left(-W^p e^{\mu}_{a}\partial_\mu x^{\underline{m}} u^{a}_{\underline{m}} + pW^{p+1}\right). \tag{1.19}$$

Then, for example, (1.15) takes the form

$$dx^{\underline{m}} = We_a u^{\underline{m}a}, \tag{1.20}$$

and (1.15) is obtained from (1.20) by gauge fixing $W = 1$. We shall encounter this situation when studying doubly supersymmetric p–branes.

On the other hand W can be eliminated from (1.19) by substituting into (1.19) the solution to its equation of motion

$$W = \frac{1}{p+1}\partial_m x^{\underline{m}} u_{\underline{m}}{}^{a} e^{m}{}_{a} \tag{1.21}$$

This results in a Weyl invariant p-brane action considered previously in [70]. Note that (1.21) does not produce any new relations between the variables, since it is just a consequence of (1.20).

1.2 Intrinsic and extrinsic geometry of the embedded surface

The geometrical approach [1, 2] implies that the world surface of a p–brane (and hence its equations of motion) is described by the pullback of the Cartan forms $\Omega^{\underline{ab}}(d)$.

To do this we should replace Eqs.(1.15), (1.17) by some equivalent system of equations on the differential forms (1.7) and the world surface vielbein e^a.

Eq. (1.17) is almost of the required type. Indeed, projecting (1.17) onto the moving frame vectors $u^a_{\underline{m}}$ and $u^i_{\underline{m}}$ we get the following two equations equivalent to (1.17)

$$\partial_m \left(e e^{ma}\right) = e e^m_b \Omega^{ba}_m, \tag{1.22}$$

$$e^m_a \Omega^{ai}_m = 0, \tag{1.23}$$

and eq. (1.15) can be replaced by its integrability conditions

$$0 = ddx^{\underline{m}} = d\left(e^a u^{\underline{m}}_a\right), \tag{1.24}$$

which ensures the possibility of finding $x^{\underline{m}}$ if e_{am} and $u^{a\underline{m}}$ are derived from (1.16), (1.17) with the Cartan forms satisfying the Maurer–Cartan equations (1.10)–(1.12).

Projecting (1.24) onto $u^a_{\underline{m}}$ and $u^i_{\underline{m}}$ we get the *metricity condition* for the induced connection on the world surface defined as the pullback of the Cartan form $\Omega^{ab} = u^a du^b$:

$$De^a \equiv T^a \equiv de^a - e_b\Omega^{ba} = 0 \tag{1.25}$$

(where $T^a_{bc}e^be^c$ is the intrinsic torsion of the surface), and the relation

$$e_a\Omega^{ai} = 0 \tag{1.26}$$

From (1.25) it follows that Ω^{ab} is completely determined in terms $e_m^{\ a}$

$$e^m_a\Omega_{mbc} \equiv e^m_a u_{\underline{m}b}\partial_m u^{\underline{m}}_c = e^m_a e^n_b \partial_{[m}e_{n]c} - e^m_b e^n_c \partial_{[m}e_{n]a} + e^m_c e^n_a \partial_{[m}e_{n]b} \tag{1.27}$$

and (1.11) is the induced Riemann curvature of the surface.

Eq. (1.26) ensures a symmetry property of the Riemann tensor (1.11) (a Bianchi identity), namely, $e^a R_a{}^b = 0$, and identifies $e_{am}\Omega^{ai}_n$ with the components K^i_{mn} of the second differential form of the surface. Indeed, by definition

$$K^i_{mn} \equiv u^i_{\underline{m}} \partial_m \partial_n x^{\underline{m}}, \tag{1.28}$$

where eq. (1.2) should be taken into account. Substituting eq.(1.15) into (1.28) taking into account (1.26) we get

$$K^i_{mn} = -e_{na} \Omega^{ai}_m, \tag{1.29}$$

Now it is easy to see that eq.(1.23) means the vanishing of the average extrinsic curvatures of the surface and, hence, defines the embedding into the flat space–time of a minimal surface

$$K^i_{mn} g^{mn} \equiv h^i = -e^m_a \Omega_m^{a\ i} = 0. \tag{1.30}$$

To complete the identification of the Cartan forms with the geometrical characteristics of the surface note that $\Omega_m^{\ ij}$ coincides with the extrinsic torsion of the surface in the target space.

The system of the Maurer–Cartan equations (1.10)–(1.12) supplemented with eqs. (1.25), (1.26) completely determines a surface $x^{\underline{m}} = x^{\underline{m}}(\xi)$ up to its rotations and displacements in the target space. When eqs. (1.10) – (1.12) are rewritten as ones determining the second quadratic form (1.28), (1.29) and the extrinsic torsion $\Omega_m^{\ ij}$, they coincide with the Codazzi, Gauss and Ricci equations, respectively. Then eqs. (1.25), (1.26) are identically satisfied.

In addition to the Codazzi, Gauss and Ricci equations the classical motion of the bosonic p–brane is characterized by Eq. (1.30) which means that the world surface is a minimal surface. This completes the description of the bosonic p–brane theory in terms of surface theory.

The geometrical approach can also be applied for studying p–branes in a curved target space, then one should use the Cartan equations for the forms in the curved manifold, which, in general, involves its torsion and curvature. Flat superspace is one of the examples of this more general situation.

Chapter 2

Towards a doubly supersymmetric geometrical approach to super–p–branes

To develop the geometrical approach in application to super–p–branes we should determine the notion of the local moving frame in a flat superspace parameterized by bosonic vector coordinate $x^{\underline{m}}$ and fermionic spinor coordinate $\theta^{\underline{\mu}}$

$$z^{\underline{M}} = (x^{\underline{m}}, \theta^{\underline{\mu}}). \tag{2.1}$$

As we will see, this naturally leads to spinor Lorentz harmonics as the fundamental constituents of the moving frame.

2.1 Spinor Lorentz harmonics as a moving frame in superspace

Let us consider a supersymmetric basis

$$\pi^{\underline{m}} \equiv dx^{\underline{m}} - id\theta\Gamma^{\underline{m}}\theta, \qquad d\theta^{\underline{\mu}} \tag{2.2}$$

in the space cotangent to the flat superspace.

Because the structure group of the flat superspace (as well as of the curved one [69]) is the double covering group $Spin(1, D-1)$ of the Lorentz group $SO(1, D-1)$, an arbitrary local frame in the flat superspace can be obtained from (2.2) by $SO(1, D-1)$ rotations, i.e.

$$\pi^{\underline{a}} \equiv \pi^{\underline{m}} u_{\underline{m}}^{\underline{a}} \qquad \psi^{\underline{\alpha}} \equiv d\theta^{\underline{\mu}} v_{\underline{\mu}}{}^{\underline{\alpha}} \tag{2.3}$$

The vector part of (2.2) is transformed by a matrix $||u_{\underline{m}}^{\underline{a}}||$ from the vector representation of the $SO(D-1)$, and the spinor part is transformed by a matrix $||v_{\underline{\mu}}{}^{\underline{\alpha}}||$ from the

spinor representation of $SO(1,D-1)$, or, strictly speaking,

$$||v_{\underline{\mu}}{}^{\underline{\alpha}}|| \quad \in \quad Spin(1,D-1). \tag{2.4}$$

Thus, the components of the new moving frame in the basis (2.2) are $u_{\underline{m}}^{\underline{a}}$ and $v_{\underline{\mu}}{}^{\underline{\alpha}}$, but since the transformations of the vector and spinor sector are characterized by the same parameters, $u_{\underline{m}}^{\underline{a}}$ and $v_{\underline{\mu}}{}^{\underline{\alpha}}$ are connected by the relation expressing the vector representation through the fundamental spinor representation of $Spin(1,D-1)$:

$$\begin{aligned} u_{\underline{m}}^{\underline{a}} &\equiv \frac{1}{2^{\nu}} v_{\underline{\mu}}^{\underline{\alpha}} (\Gamma_{\underline{m}})^{\underline{\mu\nu}} v_{\underline{\nu}}^{\underline{\beta}} (\Gamma^{\underline{a}})_{\underline{\alpha\beta}} \\ &\equiv \frac{1}{2^{\nu}} v_{\underline{\gamma}}^{\underline{\mu}} (\Gamma^{\underline{a}})^{\underline{\gamma\delta}} v_{\underline{\delta}}^{\underline{\nu}} (\Gamma_{\underline{m}})_{\underline{\mu\nu}} \end{aligned} \tag{2.5}$$

where

$$v^{-1} \equiv ||v_{\underline{\alpha}}^{\underline{\mu}}|| \quad \in \quad Spin(1,D-1) \tag{2.6}$$

is the matrix inverse to (2.4).

Hence, in superspace, the vector components of the local moving frame are naturally composed of the bosonic spinor components, the latter playing the basic role in the doubly supersymmetric [23]–[38] as well as twistor–like Lorentz–harmonic approach [39, 49, 50, 63, 40, 41, 42] (see also [54, 59])[1].

Below in this section we will present some basic properties of the spinor moving frame [39, 44, 49, 50, 40, 41, 42, 63, 43] required for further consideration (see Appendix **A** for detailes).

Due to (2.6), Eq. (2.5) can be rewritten as follows

$$u_{\underline{m}}^{\underline{a}} (\Gamma_{\underline{a}})_{\underline{\alpha\beta}} \equiv v_{\underline{\alpha}}^{\underline{\mu}} (\Gamma_{\underline{m}})_{\underline{\mu\nu}} v_{\underline{\beta}}^{\underline{\nu}}, \tag{2.7}$$

which reflects the transformation properties of the Γ–matrices with respect to the Lorentz group.

As in the bosonic case, for further description of the embedding of a super–p–brane world surface into the flat target superspace, the $SO(1,p)\times SO(D-p-1)$ invariant splitting of the composed vector moving frame $u_{\underline{m}}^{\underline{a}} = (u_{\underline{m}}^{\ a}\ ,\ u_{\underline{m}}^{\ i})$ is required. As a consequence of (2.7) this splitting is obtained by choosing an $SO(1,p)\times SO(D-p-1)$ invariant representation for the Γ–matrices

$$\Gamma^{\underline{a}} = (\Gamma^{a}, \Gamma^{i}) \tag{2.8}$$

[1] The vector moving frame variables $u_{\underline{m}}^{\underline{a}}$ are just vector Lorentz harmonics introduced by Sokatchev [51] as an extension of the concept of harmonic variables [60] to noncompact groups of space–time symmetry. For the first time the vector moving frame composed of a spinor one was introduced by Newman and Penrose [61] in application to General Relativity. In application to superparticles and superstrings vector harmonics, part of which was composed of twistor–like variables, were considered in [52] (this approach was further developed in [53]). Wiegmann [59] used the composed moving frame for the calculation of the anomalies in spinning and heterotic string.

with block–diagonal Γ^a and anti–diagonal Γ^i (for $D=11,\ p=2$ and $D=10,\ p=1$ such a representation is presented in Appendix A), and decomposing the harmonic matrix into two rectangular blocks

$$v_{\underline{\alpha}}^{\ \underline{\mu}} = (v_{\alpha q}^{\ \ \underline{\mu}}\ ,\ v_{\dot{q}}^{\alpha\underline{\mu}}), \tag{2.9}$$

where the index α corresponds to the spinor representation of $SO(1,p)$ and $(p,\ \dot{q})$ stand for two (in general non–equivalent) spinor representations of $SO(D-p-1)$. Then, the relation (2.7) splits into three $SO(1,p)\times SO(D-p-1)$ invariant relations

$$\delta_{qp}(\gamma_a)_{\alpha\beta}u_{\underline{m}}^{\ a} = v_{\alpha q}\Gamma_{\underline{m}}v_{\beta p}, \tag{2.10}$$

$$\delta_{\dot{q}\dot{p}}(\gamma_a)^{\alpha\beta}u_{\underline{m}}^{a} = v_{\dot{q}}^{\alpha}\Gamma_{\underline{m}}v_{\dot{p}}^{\beta}, \tag{2.11}$$

$$\delta_{\beta}^{\alpha}\gamma_{q\dot{p}}^{i}u_{\underline{m}}^{i} = v_{\alpha q}\Gamma_{\underline{m}}v_{\dot{p}}^{\beta}, \tag{2.12}$$

where $\gamma_a^{\alpha\beta}$ and $\gamma_{q\dot{p}}^{i}$ are the $SO(1,p)$ and $SO(D-p-1)$ γ–matrices, respectively (see Appendix **A** for the D=11, p=2 case).

We see that in (2.10)–(2.12) $(v_{\alpha q}^{\underline{\mu}}, v_{\dot{q}}^{\alpha\underline{\mu}})$, as well as $(u_{\underline{m}}^{a}, u_{\underline{m}}^{i})$, are determined up to the local $SO(1,p)\times SO(D-p-1)$ transformations and can be identified with spinor Lorentz harmonics parametrizing the coset space $\frac{SO(1,D-1)}{SO(1,p)\times SO(D-p-1)}$. Note that in contrast to the splitting of vectors that of the spinors results in multiplicative structure of the $SO(1,p)\times SO(D-p-1)$ spinor indices.

This basic notion on the local moving frame in superspace is sufficient for developing the geometrical approach to super–p–branes.

In the conventional formulation of super–p–branes [7, 8, 9, 13, 10] one considers the embedding of a bosonic world surface spanned by a p–brane moving in a target superspace. In the approach under consideration this embedding can be described by an action which is the sum of the conventional Wess–Zumino term [6, 7, 8, 9] plus an analog of Eq. (1.4) where $\partial_m x^{\underline{m}}(\xi)$ is replaced with the $\Pi_n^{\ \underline{m}}$ component of the supersymmetric Cartan form $\Pi^{\ \underline{m}} = dx^{\underline{m}} - id\theta\Gamma^{\underline{m}}\theta$ and $u_{\underline{m}}^{a}(\xi)$ is composed of $v_{\underline{\alpha}}^{\ \underline{\mu}}(\xi)$ (Eq.(2.10)) [40, 41, 42, 43]. Then, to derive equations of motion of $v_{\underline{\alpha}}^{\ \underline{\mu}}(\xi)$, one should, as in the bosonic case (eqs. (1.6)), consider the variations of $v_{\underline{\alpha}}^{\ \underline{\mu}}(\xi^m)$, which do not violate the condition (2.4), (2.5). These variations are:

$$\delta v_{\underline{\alpha}}^{\ \underline{\mu}} = -O_{\underline{\alpha}}^{\ \underline{\beta}}v_{\underline{\beta}}^{\ \underline{\mu}} = -\frac{1}{4}O^{\underline{ab}}(\Gamma_{\underline{ab}})_{\underline{\alpha}}^{\ \underline{\gamma}}v_{\underline{\gamma}}^{\ \underline{\mu}}.$$

And

$$dv_{\underline{\alpha}}^{\ \underline{\mu}} = -\frac{1}{4}\Omega^{\underline{ab}}(d)(\Gamma_{\underline{ab}})_{\underline{\alpha}}^{\ \underline{\gamma}}v_{\underline{\gamma}}^{\ \underline{\mu}}. \tag{2.13}$$

(compare with (1.6), (1.7)).

The studying of the constraints and the equations of motion of super–p–branes in the Lorentz harmonic formulation was performed in [40, 41, 42, 43], so we only note that this formulation can be regarded as a component version of a doubly supersymmetric p–brane model [23]–[38], and proceed with developing the geometrical approach to the latter.

2.2 Geometrodynamical condition, twistor constraint and geometrical framework for the description of super–p–branes

In the doubly supersymmetric formulation of super–p–branes [23]–[38], their dynamics is described by embedding the world supersurface

$$z^M \equiv (\xi^m, \eta^{\alpha q}),$$

into the target superspace (2.1), which is further considered to be flat,

$$z^{\underline{M}} = Z^{\underline{M}}(z^M).$$

Note that supersurfaces under consideration the number of the Grassmann directions is half of the number of the target superspace Grassmann directions. An intrinsic world supersurface geometry is assumed to be characterized by torsion constraints [37, 38]

$$\begin{array}{rcl} T^a \equiv DE^a \equiv dE^a - E^b w_b{}^a & = & -iE^{\alpha q}E^{\beta}_{q}\gamma^a_{\alpha\beta}, \\ T^{\alpha q} \equiv DE^{\alpha q} & \equiv & dE^{\alpha q} - E^{\beta q} w_\beta{}^\alpha + E^{\alpha p}A_p{}^q \\ & & = -E^a E^{\beta p} T_{a\beta p}{}^{\alpha q} - E^a E^b T_{ab}{}^{\alpha q}, \end{array} \tag{2.14}$$

where

$$E^a \equiv dz^M E^{\ a}_M \qquad and \qquad E^{\alpha q} \equiv dz^M E^{\ \alpha q}_M \tag{2.15}$$

are the vector and spinor world supersurface vielbein forms, and $w_b{}^a$, $A_p{}^q$ are the components of an $SO(1,p) \times SO(D-p-1)$ connection.

We should stress that the only essential torsion constraint in (2.14) is $T_{\alpha q\ \beta p}{}^a = -2i\delta_{qp}\gamma^a_{\alpha\beta}$ ensuring flat supersurface limit. The other torsion constraints are obtained by solving for the Bianchi identities and redefining vielbeins and connections.

Eqs. (2.14) imply, in particular, the following anticommutation relations for supercovariant spinor derivatives

$$\{D_{\alpha q}\ ,\ D_{\beta p}\} = 2i\delta_{qp}\gamma^a{}_{\alpha\beta}D_a + R_{\alpha q\ \beta p}, \tag{2.16}$$

where $R_{\alpha q\ \beta p}$ are components of intrinsic $SO(1,p) \times SO(D-p-1)$ curvature

$$SO(1,p): \qquad R^{ab} = dw^{ab} - w^{ac}w_c{}^b, \qquad R^{\alpha\beta} \sim R^{ab}\gamma^{\alpha\beta}_{ab}, \tag{2.17}$$

$$SO(D-p-1): \qquad R^{ij} = dA^{ij} + A^{ik}A^{kj}, \qquad R^{pq} \sim R^{ij}\gamma_{ij}{}^{pq}. \tag{2.18}$$

Note that at least for superstrings (p=1) and supermembranes (p=2) the constraints (2.14) are intact under super–Weyl transformations of the supervielbeins

$$\hat{E}^a = W^2 E^a, \qquad \hat{E}^{\alpha q} = WE^{\alpha q} - iE^b\gamma_b^{\alpha\beta}D^q_\beta W \tag{2.19}$$

and corresponding transformations of the connection forms. This property will be used below for studying a D=11 supermembrane and D=10 superstrings.

Let us consider the pullback of the one–forms (2.2) onto the world supersurface

$$\Pi^{\underline{m}} \equiv dX^{\underline{m}} - id\Theta\Gamma^{\underline{m}}\Theta = E^{\alpha q}\Pi^{\underline{m}}_{\alpha q} + E^{a}\Pi^{\underline{m}}_{a}, \tag{2.20}$$

$$d\Theta^{\underline{\mu}} = E^{\alpha q}D_{\alpha q}\Theta^{\underline{\alpha}} + E^{a}D_{a}\Theta^{\underline{\mu}}. \tag{2.21}$$

where, in view of (2.16),

$$2i\delta_{qp}\gamma^{a}_{\alpha\beta}D_{a}\Theta^{\underline{\mu}} = D_{\alpha q}D_{\beta p}\Theta^{\underline{\mu}} + D_{\beta p}D_{\alpha q}\Theta^{\underline{\mu}}, \tag{2.22}$$

The twistor–like bosonic superfield $D_{\alpha q}\Theta^{\underline{\mu}}$ plays the basic role in the development of the doubly supersymmetric geometrical approach.

The embedding of a super–p–brane world supersurface is specified by a geometrodynamical condition [23]–[38], which requires the vanishing of $\Pi^{\underline{m}}$ along the Grassmann world supersurface directions:

$$\Pi^{\underline{m}}_{\alpha q} \equiv D_{\alpha q}X^{\underline{m}} - iD_{\alpha q}\Theta\Gamma^{\underline{m}}\Theta = 0. \tag{2.23}$$

The integrability condition for (2.23)

$$\delta_{qp}\gamma^{a}{}_{\alpha\beta}\Pi_{a}{}^{\underline{m}} = D_{\alpha q}\Theta\Gamma^{\underline{m}}D_{\beta p}\Theta \tag{2.24}$$

is called "twistor constraint".

Eq.(2.24) looks very much like Eq. (2.10) which relates the spinor and vector Lorentz harmonics. So we argue that $D_{\alpha q}\Theta^{\underline{\mu}}$ can be identified (up to a scalar superfield factor W) with $v_{\alpha q}^{\underline{\mu}}$ (2.5), (2.7), and $\Pi^{\underline{m}}_{a}$, can be identified with $u^{\underline{m}}_{a}$ (2.10), (up to the square W^2 of the same factor):

$$D_{\alpha q}\Theta^{\underline{\mu}} = Wv_{\alpha q}^{\underline{\mu}}, \tag{2.25}$$

$$\Pi^{\underline{m}}_{a} = D_{a}X^{m} - D_{a}\Theta\Gamma^{\underline{m}}\Theta = W^{2}u^{\underline{m}}_{a}, \tag{2.26}$$

For a $D = 11$, N=1 supermembrane a direct proof of (2.25), (2.26) is presented in the Appendix **B**.

Thus, the spinor moving frame ($\equiv$ Lorentz harmonics), which is the generalization of the Cartan moving frame to the case of superspace, naturally appears in the doubly supersymmetric p–brane formulation.

Eq. (2.26)[2] can be regarded as the supersymmetric counterpart of eq. (1.3), and eq. (2.25) is a "square root" of (2.26).

In view of (2.23), (2.22), (2.25) and (2.26) the one-forms (2.20), (2.21) are expressed as follows

$$\Pi^{\underline{m}} = E^{a}W^{2}u^{\underline{m}}_{a}, \tag{2.27}$$

[2] The leading component of this equation appears as an equation of motion in the twistor–like Lorentz harmonic super–p–brane formulation [42, 43].

$$d\Theta^{\underline{\mu}} = WE^{\alpha q} v_{\alpha q}^{\ \underline{\mu}} + E^a D_a \Theta^{\underline{\mu}}. \tag{2.28}$$

From (2.27), (2.28) we conclude that an induced metric $\Pi^{\underline{m}}\Pi_{\underline{m}}$ on the supersurface coincides with an intrinsic metric $E^a E_a$ up to the scale factor W^4. Thus W plays the role of a rescaling factor of the intrinsic metric.

To study the properties of the supersurface embedding it is necessary to consider the integrability conditions for eqs. (2.27), (2.28) with taking into account (2.23)

$$d\Pi^{\underline{m}} = -id\Theta\Gamma^{\underline{m}} d\Theta, \tag{2.29}$$

$$dd\Theta^{\underline{\mu}} = 0. \tag{2.30}$$

Eqs. (2.29), (2.30) are the pullback of the Maurer–Cartan equations for supertranslations in the flat superspace [69] (they should not be confused with the Maurer–Cartan equations (1.10) – (1.12) for $SO(1, D-1)$).

Substituting into (2.29), (2.30) the expression for $\Pi^{\underline{m}}$ and $d\Theta^{\underline{\mu}}$ in terms of the harmonics ((2.27), (2.28)) and projecting onto the $SO(1,p) \times SO(d-p-1)$ directions we get

$$d(W^2 E^a) - (W^2 E^b)\Omega_b^{\ a} = -id\Theta\Gamma^{\underline{m}} d\Theta u_{\underline{m}}^a \equiv T^a_{\text{ind}}, \tag{2.31}$$

$$(W^2 E^b)\Omega_b^{\ i} = id\Theta\Gamma^{\underline{m}} d\Theta u_{\underline{m}}^i, \tag{2.32}$$

$$d(WE^{\alpha p}) - (WE^{\beta q})\Omega_{\beta q}^{\ \ \alpha p} = T^{\alpha p}_{\text{ind}}, \tag{2.33}$$

$$(WE^{\beta q})\Omega_{\beta q}^{\ \ \alpha\dot{p}} = T^{\alpha\dot{p}}_{\text{ind}}. \tag{2.34}$$

Eqs. (2.31), (2.33) (see (2.13) for the Ω–notation) define the components Ω^{ab}, $\Omega_{\alpha p}^{\ \ \beta q} = \Omega_\alpha^\beta \delta_q^p - \Omega_p^{\ q}\delta_\alpha^\beta$ of induced connection on the supersurface, with T^a_{ind}, $T^{\alpha p}_{\text{ind}}$ being the components of induced torsion. (Since the explicit form of $T^{\alpha p}_{\text{ind}}$, $T^{\alpha\dot{p}}_{\text{ind}}$ is rather complicated, we do not present it here). The Ω–forms in (2.31)–(2.34) satisfy the Maurer–Cartan equations (1.10)–(1.12), and eqs. (2.32), (2.34) ensure symmetry properties of the Riemann tensor in the presence of induced torsion.

One can see that, in general, the supersurface geometry induced by the embedding under consideration differs from the intrinsic geometry defined by eqs. (2.14), and the set of equations (2.23), (2.25), (2.26), (2.31)–(2.34) and the Maurer–Cartan equations (1.10)–(1.12) relates the two kinds of geometry. In particular, from (2.31), (2.14) it follows that the spinor-spinor components of $W^{-2}T^a_{ind}$ coincide with that of T^a.

To get all the consequences of eqs. (2.29), (2.30) (or (2.31)–(2.34)), and eqs. (1.10)–(1.12) one has to solve for their components in the basis

$$E^{\alpha q}E^{\beta p}\ , \qquad E^a E^{\beta p}\ , \qquad E^a E^b\ .$$

It turns out, however, that for a supersurface with $n > 1$ only equations corresponding to the spinor–spinor components are independent (see Appendix **C**)[3]. This means that

[3] Of course, the choice of the independent relations is not unique.

all other consequences can be derived by taking the spinor derivatives of the spinor-spinor components and using (2.16). For example, the independent consistency equations contained in (2.29), (2.30) are (2.22), (2.24).

To compare the induced geometry with the intrinsic one it is convenient to introduce covariant objects reflecting the difference between induced and intrinsic connection:

$$\Omega^{ab}(D) = \Omega^{ab}(d) - w^{ab}, \tag{2.35}$$

$$\Omega^{ij}(D) = \Omega^{ij}(d) - A^{ij}, \tag{2.36}$$

$$\Omega^{ai}(D) = \Omega^{ai}(d). \tag{2.37}$$

Note that in the bosonic case (Chapter 1), where the induced and intrinsic geometry coincide, $\Omega\ ^{ab}(D) = 0$.

In terms of (2.35)–(2.37) the Maurer–Cartan equations take the form

$$D^{ind}\Omega^{ai} = D\Omega^{ai} - \Omega^{a}{}_{b}(D)\Omega^{bi} + \Omega^{aj}\Omega^{ji}(D) = 0, \tag{2.38}$$

$$R^{ab}_{ind} = D\Omega^{ab}(D) - \Omega^{a}{}_{c}(D)\Omega^{cb}(D) + R^{ab} = \Omega^{ai}\Omega^{bi}, \tag{2.39}$$

$$R^{ij}_{ind} = D\Omega^{ij}(D) + \Omega^{ij'}(D)\Omega^{j'j}(D) + R^{ij} = \Omega^{ai}\Omega_{a}{}^{j}. \tag{2.40}$$

The independent (spinor–spinor) components of (2.38)–(2.40) are

$$\begin{aligned} -2i\delta_{qp}\gamma^{b}_{\alpha\beta}\Omega_{b}{}^{ai} &= D_{\alpha q}\Omega^{ai}_{\beta p} + \Omega_{\alpha q}{}^{a}{}_{b}\Omega_{\beta p}{}^{bi} - \Omega_{\alpha q}{}^{aj}\Omega^{ji}_{\beta p} \\ &\quad +((\alpha q) \leftrightarrow (\beta p)), \end{aligned} \tag{2.41}$$

$$\begin{aligned} -2i\delta_{qp}\gamma^{c}_{\alpha\beta}\Omega_{c}{}^{ab} &= D_{\alpha q}\Omega^{ab}_{\beta p} + \Omega_{\alpha q}{}^{a}{}_{c}\Omega_{\beta p}{}^{cb} + \Omega_{\alpha q}{}^{aj}\Omega^{bj}_{\beta p} + R^{ab}_{\alpha q\beta p} \\ &\quad +((\alpha q) \leftrightarrow (\beta p)), \end{aligned} \tag{2.42}$$

$$\begin{aligned} -2i\delta_{qp}\gamma^{c}_{\alpha\beta}\Omega_{c}{}^{ij} &= D_{\alpha q}\Omega^{ij}_{\beta p} - \Omega_{\alpha q}{}^{ik}\Omega_{\beta p}{}^{kj} - \Omega_{\alpha q}{}^{ai}\Omega_{\beta pa}{}^{j} + R^{ij}_{\alpha q\beta p} \\ &\quad +((\alpha q) \leftrightarrow (\beta p)), \end{aligned} \tag{2.43}$$

Eqs.(2.38)–(2.40) (or (2.41) – (2.43)), with taking into account (2.31)–(2.34), can be regarded as supersymmetric analogs of Codazzi, Gauss and Ricci equations.

2.3 Minimal supersurface embedding into flat superspace

We have seen in the bosonic case that the equations of motion of a p–brane determine minimal embedding of the world surface, and the minimal surface is characterized by the traceless second fundamental form (eqs.(1.28)–(1.30)).

In the doubly supersymmetric case we shall also assume that the equations of motion of a super–p–brane determine a minimal embedding of the world supersurface (and vice versa), which is characterized by the vanishing trace of a supersymmetric counterpart of the bosonic second fundamental form.

An appropriate $SO(1,p)\times SO(D-p-1)$–valued bilinear form on the supersurface has the following components

$$K^{i}_{AB}=(K^{i}_{ab},K^{i}_{\alpha pb},K^{i}_{\alpha p\beta q}), \tag{2.44}$$

$$K^{\gamma\dot r}_{AB}=(K^{\gamma\dot r}_{ab},K^{\gamma\dot r}_{\alpha pb},K^{\gamma\dot r}_{\alpha p\beta q}), \tag{2.45}$$

and is symmetric with respect to the permutations of the vector–vector and vector–spinor indices and antisymmetric with respect to the permutations of the pairs of spinor indices $(\alpha p,\beta q)$.

Since the structure group in the supersurface tangent space is $SO(1,p)\times SO(D-p-1)$, each component in (2.44), (2.45) transforms independently and, thus, can be regarded as an independent supersymmetric bilinear form. For describing the embedding in question we suppose that it is sufficient to determine the K^{i}_{ab} and $K^{\gamma\dot r}_{\alpha p\beta q}$ component of (2.44), (2.45) as supersymmetric analogs of the second fundamental form (1.28), (1.29), and take

$$K^{i}_{ab}=2\left(D_{\{a}\Pi^{\underline m}_{b\}}\right)u^{i}_{\underline m}=-2W^{2}\Omega_{\{ab\}}{}^{i}, \tag{2.46}$$

$$K^{\gamma\dot r}_{\alpha p\beta q}=2\left(D_{[\alpha p}D_{\beta q]}\Theta^{\underline\mu}\right)v^{\gamma\dot r}_{\underline\mu}=-W\Omega^{ai}_{\alpha q}(\gamma_a)^{\gamma}_{\beta}(\gamma_i)^{\dot r}_{q}-((\alpha p)\leftrightarrow(\beta q)). \tag{2.47}$$

As in the bosonic case, (2.46), (2.47) are expressed through the components of $\Omega^{ai}(d)$.

We assume that the minimal embedding of the supersurface into the flat superspace is characterized by (2.46), (2.47) with vanishing traces:

$$K^{ai}_{a}=0=\Omega^{ai}_{a}=i\frac{1}{p2^{[\frac{p}{2}]}}\left(D_{\alpha q}\left((\gamma_a)^{\alpha\beta}\Omega^{ai}_{\beta q}\right)+\Omega^{a}_{\alpha qb}(\gamma_a)^{\alpha\beta}\Omega^{bi}_{\beta q}-(\gamma_a)^{\alpha\beta}\Omega^{aj}_{\alpha q}\Omega^{ji}_{\beta q}\right), \tag{2.48}$$

$$K^{\alpha p,\gamma\dot r}_{\alpha p}=0=(\gamma_a)^{\gamma\alpha}\Omega^{ai}_{\alpha p}(\gamma_i)^{p\dot r} \tag{2.49}$$

where the r.h.s. of (2.48) follows from eq. (2.41).

The sufficient condition for (2.49) to hold is

$$\gamma^{a}{}_{\alpha}{}^{\beta}\Omega_{\beta q}{}^{ai}=0. \tag{2.50}$$

And it is just this condition which follows from equations of motion of the super–p–brane.

Indeed, the appropriate form of the equations of motion of $\Theta^{\underline\mu}(z^M)$ can be obtained by a superfield generalization of the equation of motion of the field $\theta^{\underline\mu}(\xi^m)$ arisen in the Lorentz harmonic super–p–brane formulation [42, 43]:

$$\gamma^{a}_{\alpha\beta}D_a\Theta^{\underline\mu}v^{\beta\dot q}_{\underline\mu}=0. \tag{2.51}$$

(Recall that $v_{\alpha q}{}^{\underline{\mu}} v_{\underline{\mu}}^{\gamma\dot{r}} = 0$).

Using eqs. (2.22), (2.25), (2.13) one can represent the l.h.s. of (2.51) as

$$2i\delta_{qp}(\gamma^a)_{\alpha\beta} D_a \Theta^{\underline{\mu}} v_{\underline{\mu}}^{\beta\dot{r}} = (\gamma_a)_\alpha^\beta \Omega^{ai}_{\beta q} (\gamma_i)_p^{\dot{r}}, \tag{2.52}$$

from which we conclude that eq. (2.51) holds when $\Omega^{ai}_{\beta q}$ satisfies eq. (2.50) and vice versa. In view of (2.46), (2.48) this also leads to the equations of motion of $X^{\underline{m}}$ provided $\Omega^{ab}_{\alpha p}$ is restricted by consistency conditions to be

$$\Omega^{ab}_{\alpha p} \gamma_a^{\alpha\beta} = \varphi_{\alpha p} \gamma^{b\alpha\beta}, \tag{2.53}$$

where $\varphi_{\alpha p}$ is a spinorial superfield (we shall encounter this situation below).

Note that (2.48) specifies the $X^{\underline{m}}$ components along the directions orthogonal to the world supersurface. The $X^{\underline{m}}$ components along the directions tangent to the supersurface can be eliminated by fixing a gauge with respect to the local symmetries.

In the next two chapters we shall consider in more detail some particular features of the world supersurface embedding in the case of D=11 supermembranes and D=10 superstrings. For instance, eq. (2.53) and the equation of motion (2.51) of a D=11, N=1 supermembrane and D=10, N=II superstrings will appear as a consequence of the geometrodynamical condition (2.23).

Chapter 3

N=1 supermembrane in D=11

In the previous chapter we have obtained the system of equations determining an embedding of a world supersurface into the flat target superspace and relating intrinsic and induced geometry on the supersurface.

Below we shall study these equations in application to a supermembrane (i.e. $p=2$) in N=1, D=11 target superspace possessing $n=D-p-1=8$ world sheet supersymmetries [37]. In particular, we will see that the equation of motion (2.51) is among the consequences of the geometrodynamical condition (2.23), and that the difference between the spinor components of the intrinsic and induced $SO(1,2)\times SO(8)$ connection of the world surface (eqs. (2.35), (2.36)) is due to the presence of the scale factor $W(\xi,\eta)$.

To show this let us consider one of the independent equations of the integrability condition (2.30), namely (2.22), and take into account (2.25) (the latter being the consequence of the twistor constraint (2.24) and, hence, of the geometrodynamical condition (2.23)):

$$2i\delta_{qp}\gamma^{a}_{\alpha\beta}D_a\Theta^{\underline{\mu}}=(D_{\alpha q}W)v^{\underline{\mu}}_{\beta p}+WD_{\alpha q}v^{\underline{\mu}}_{\beta p}+((\alpha q)\leftrightarrow(\beta p)). \tag{3.1}$$

For further consideration one should make use of the relation (2.12) and that of Appendix A allowing one to express the covariant differential of $v^{\underline{\mu}}_{\alpha q}$ in terms of $\Omega^{\underline{ab}}(D)$ (eqs. (2.35)–(2.37)):

$$Dv^{\underline{\mu}}_{\alpha q}=\frac{i}{4}\epsilon_{abc}\Omega^{ab}(D)\gamma^{c\ \beta}_{\ \alpha}v^{\underline{\mu}}_{\beta q}+\frac{1}{4}\Omega^{ij}(D)\gamma^{ij}_{qp}v^{\underline{\mu}}_{\alpha p}-\frac{1}{4}\Omega^{ai}(D)\gamma_{a\ \alpha\beta}\gamma^{i}_{q\dot{p}}v^{\beta\underline{\mu}}_{\dot{p}}. \tag{3.2}$$

Projecting (3.1) onto the "orthogonal" directions $v_{\underline{\mu}\gamma\dot{r}}, v_{\underline{\mu}\gamma r}$ one gets

$$\begin{aligned}2i\delta_{qp}\gamma^{a}{}_{\alpha\beta}(D_a\Theta^{\underline{\mu}})v_{\underline{\mu}\gamma r} &= D_{\alpha q}W\epsilon_{\beta\gamma}\delta_{pr}+\tfrac{iW}{4}\epsilon_{abc}\Omega^{\ ab}_{\alpha q}\gamma^{c}_{\beta\gamma}\delta_{pr}+\tfrac{W}{4}\Omega^{\ ij}_{\alpha q}\epsilon_{\beta\gamma}\gamma^{ij}_{pr}\\ &\qquad +((\alpha q)\leftrightarrow(\beta p)),\end{aligned} \tag{3.3}$$

$$2i\delta_{qp}\gamma^{a}{}_{\alpha\beta}(D_a\Theta^{\underline{\mu}})v_{\underline{\mu}\gamma\dot{r}}=\frac{1}{4}W\Omega^{\ ai}_{\alpha q}\gamma_{a\ \beta\gamma}\gamma^{i}_{p\dot{r}}+((\alpha q)\leftrightarrow(\beta p)), \tag{3.4}$$

from which we shall derive eq. (2.51), (2.53) and which will allow us to express $\Omega^{\ ab}_{\alpha q}\equiv\Omega^{\ ab}(D_{\alpha q})$, $\Omega^{\ ij}_{\alpha q}\equiv\Omega^{\ ij}(D_{\alpha q})$ in terms of $D_{\alpha q}W$.

Now recall that for the supermembrane under consideration the constraints (2.14) are invariant under the super–Weyl transformations (2.19) of the supervielbeins and the corresponding transformations of the intrinsic $SO(1,2)\times SO(8)$ connection

$$
\begin{aligned}
\hat{\omega}^{ab} \; &= \omega^{ab} + 2W^{-1}(D^b W E^a - D^a W E^b) + \tfrac{1}{2} W^{-2} \epsilon^{abc} E_c D_{\alpha q} W D^{\alpha q} W \\
&\qquad +2iW^{-1}\epsilon^{abc}\gamma_{c\alpha}^{\ \beta} E^{\alpha q} D_{\beta q} W, \\
\hat{A}^{ij} &= A^{ij} + \tfrac{1}{W}\gamma^{ij}_{qp} D_{\alpha p} W + \ldots
\end{aligned}
\tag{3.5}
$$

(where dots denote insignificant terms). This allows one to put W=1 in (3.1), (3.4), (3.3) without violating the constraints.

3.1 Relation between intrinsic and induced connection

Let us consider eq. (3.3) with $W=1$.

First of all note that the SO(8) irreps **56** and **160** cannot be contained in $\Omega_{\alpha q}^{\ ij}\gamma^{ij}_{pr}$ (due to the structure of the other terms in (3.3)), hence its general form is

$$
\Omega_{\alpha q}^{\ ij}\gamma^{ij}_{pr} = \phi_{\alpha [p}\delta_{r] q}. \tag{3.6}
$$

Then, let us decompose the r.h.s and the l.h.s. of (3.3) onto the SO(1,2) irreducible parts as follows

$$
\text{in l.h.s.} \qquad 2i\gamma^{a}_{\ \alpha\beta} D_a \Theta^{\underline{\mu}} v_{\underline{\mu}\ \gamma r} \equiv \psi_{\{\alpha\beta\gamma\} r} + \frac{2}{3}\epsilon_{\gamma\{\alpha}\psi_{\beta\} r} \tag{3.7}
$$

$$
\text{in r.h.s.} \qquad \frac{i}{4}\epsilon_{abc}\Omega_{\alpha q}^{\ \ ab}\gamma^{c}_{\ \beta\gamma} \equiv \kappa_{\{\alpha\beta\gamma\} q} + \frac{2}{3}\epsilon_{\alpha\{\beta}\kappa_{\gamma\} q}. \tag{3.8}
$$

The part of (3.3) being completely symmetric in $\{\alpha\beta\gamma\}$ has the form

$$
\delta_{qp}\psi_{\{\alpha\beta\gamma\} r} = 2\kappa_{\{\alpha\beta\gamma\}\ \{q}\delta_{p\} r}. \tag{3.9}
$$

Putting $q=p\neq r$ for any r we get from Eq. (3.9)

$$
\psi_{\{\alpha\beta\gamma\} r} = 0 = \kappa_{\{\alpha\beta\gamma\} q}, \tag{3.10}
$$

and, hence,

$$
\frac{i}{4}\epsilon_{abc}\Omega_{\alpha q}^{\ \ ab}\gamma^{c}_{\ \beta\gamma} = \frac{i}{6}\epsilon_{abc}\epsilon_{\alpha\{\beta}\gamma^{c\delta}_{\ \ \gamma\}}\Omega_{\delta q}^{\ \ ab}. \tag{3.11}
$$

Then, contracting eq. (3.3) with $\epsilon^{\alpha\beta}$ and using (3.6) we obtain

$$
\frac{1}{4}\Omega_{\alpha q}^{\ ij}\gamma^{ij}_{pr} = \frac{1}{4}\phi_{\alpha[p}\delta_{r]q} = -\frac{i}{2}\epsilon_{abc}\Omega_{\delta [p}^{\ \ ab}\gamma^{c\delta}_{\ \ \alpha}\delta_{r]q}. \tag{3.12}
$$

Substituting eq. (3.12) back into eq. (3.3) and taking into account relations found above upon some manipulation with indices we finally get

$$
\Omega^{\ ab}(D_{\alpha q}) = \Omega^{\ ij}(D_{\alpha q}) = (D_a\Theta^{\underline{\mu}}) v_{\underline{\mu}\gamma r} = 0. \tag{3.13}
$$

Thus the pull back of $d\Theta^{\underline{\mu}}$ is expressed in terms of the spinor Lorentz harmonics and $\psi_{\{\alpha\beta\gamma\}\dot{q}}$ as follows:

$$d\Theta^{\underline{\mu}} = E^{\alpha q} v_{\alpha q}{}^{\underline{\mu}} - E^{a}\frac{1}{2}\gamma_{a}^{\alpha\beta}\psi_{\{\alpha\beta\gamma\}\dot{q}} v_{\dot{q}}^{\gamma\underline{\mu}}. \tag{3.14}$$

If one performs the inverse super–Weyl transformation the r.h.s. of (3.13) will become nonzero:

$$\frac{i}{8}W\epsilon_{abc}\Omega^{\ ab}(D_{\alpha q})\gamma^{c}_{\beta\gamma} = \epsilon_{\alpha\{\beta}D_{\gamma\}q}W \qquad or \qquad \Omega^{\ ab}(D_{\alpha q}) = 2i\epsilon^{abc}\gamma_{c\ \alpha}^{\ \ \beta}\frac{1}{W}D_{\beta q}W, \tag{3.15}$$

$$\Omega^{\ ij}(D_{\alpha q}) = \gamma^{ij}_{qp}\frac{1}{W}D_{\alpha p}W, \tag{3.16}$$

and

$$i\gamma^{a}_{\alpha\beta}(D_{a}\Theta^{\underline{\mu}})v_{\underline{\mu}\gamma r} = 2D_{\{\alpha r}W\epsilon_{\beta\}\gamma}. \tag{3.17}$$

We see that the difference between the $E^{\alpha q}$ components of the induced and the intrinsic $SO(1,2)\times SO(8)$ connection (eqs. (3.15), (3.16)) is due to the superfield W, and $\Omega^{ai}(D_{\alpha q})$ is restricted to be of the form (3.28). Further on, eqs. (2.41)–(2.43) allow one to determine the E^{a} components of $\Omega^{\underline{ab}}(D)$ in terms of their spinor components. Note that (3.15) is a particular case of (2.53).

Because of the super-Weyl invariance of the d=2+1, n=8 supergravity constraints, the superfield W can be gauged away of the theory, so that the intrinsic and induced geometry on the world supersurface coincide at least for the spinor components of the connections. Note that then the vector component of Ω^{ai} satisfies the condition similar to (1.26) in the bosonic case

$$E_{a}E^{b}\Omega^{ai}_{b} = 0$$

(which follows from (2.32), (3.14)) and can be regarded as the vector part (2.46) of the supersymmetric analog of the second fundamental form.

The difference $\Omega^{ab}(D_{c})$ between the vector components of intrinsic and induced spin connection is due to nonzero components T^{a}_{bc} of induced torsion (2.31) while intrinsic T^{a}_{bc} was chosen to be zero (2.14). To completely identify the two connections one should redefine ω^{ab}_{c} in such a way that $\Omega^{ab}(D_{c}) = 0$, of course then T^{a}_{bc} will become non–zero in the constraints (2.14).

3.2 Minimal embedding of the supermembrane world surface.

To find the restrictions on $\Omega^{ai}_{\alpha q}$ and $D_{a}\Theta^{\underline{\mu}}$ which follow from eq.(3.4) we decompose the r.h.s. and the l.h.s. of (3.4) onto the irreducible representations of $SO(1,2)$:

$$\text{in l.h.s.} \qquad 2i\gamma^{a}{}_{\alpha\beta}D_{a}\Theta^{\underline{\mu}}v_{\underline{\mu}\ \gamma\dot{r}} \equiv \psi_{\{\alpha\beta\gamma\}\dot{r}} + \frac{2}{3}\epsilon_{\gamma\{\alpha}\psi_{\beta\}\dot{r}} \tag{3.18}$$

$$\text{in r.h.s.} \qquad \frac{1}{4}\gamma^{a}{}_{\alpha\beta}\Omega_{\gamma q}{}^{ai} \equiv \kappa^{i}_{\{\alpha\beta\gamma\}q} + \frac{2}{3}\epsilon_{\gamma\{\alpha}\kappa^{i}_{\beta\}q}, \tag{3.19}$$

where {...} and [...] denote, respectively, the symmetrization and antisymmetrization of the indices enclosed.

Comparing (3.18) with (3.19) in (3.4) we have

$$\delta_{qp}\psi_{\{\alpha\beta\gamma\}\dot{r}} = 2\kappa^{i}_{\{\alpha\beta\gamma\}\{q}\gamma^{i}_{p\}\dot{r}}, \tag{3.20}$$

$$\delta_{qp}\epsilon_{\gamma\{\alpha}\psi_{\beta\}\dot{r}} = -2\epsilon_{\gamma\{\alpha}\kappa^{i}_{\beta\}\{q}\gamma^{i}_{p\}\dot{r}}, \tag{3.21}$$

$$0 = \epsilon_{\alpha\beta}\kappa^{i}_{\gamma[q}\gamma^{i}_{p]\dot{r}}. \tag{3.22}$$

Substituting the solution to (3.21)

$$\kappa_{\alpha q}{}^{i} = -2\gamma^{i}_{q\dot{q}}\psi_{\alpha\dot{q}} \tag{3.23}$$

into (3.22) we get

$$\psi_{\alpha\dot{q}}\tilde{\gamma}^{i}_{\dot{q}[q}\gamma^{i}_{p]\dot{r}} = 0, \tag{3.24}$$

which has only the trivial solution

$$\psi_{\alpha\dot{q}} \equiv 2i\gamma^{a}{}_{\alpha}{}^{\beta}D_{a}\Theta^{\underline{\mu}}v_{\underline{\mu}\ \beta\dot{q}} = 0. \tag{3.25}$$

Eq. (3.25) is just the equation of motion of the Grassmann superfield $\Theta^{\underline{\mu}}$. Due to (3.23) we also have

$$\kappa_{\alpha q}{}^{i} \equiv \frac{1}{4}\gamma^{a}{}_{\alpha}{}^{\beta}\Omega_{\beta q}{}^{ai} = 0, \tag{3.26}$$

and, as a consequence of the supersymmetric Codazzi equation (2.41) (or (2.48)) and the condition (3.15)

$$\Omega_{a}{}^{ai} \equiv E_{a}{}^{M}\Omega^{ai}_{M} = 0, \tag{3.27}$$

which is, in fact, the equation of motion for $X^{\underline{m}}$ superfield.

From eqs. (3.19), (3.26) it follows that

$$\frac{1}{4}\Omega^{ai}_{\alpha q} = i\gamma^{a\ \beta\gamma}\psi_{\{\alpha\beta\gamma\}\dot{q}}\tilde{\gamma}^{i}_{\dot{q}q}. \tag{3.28}$$

On the contrary, if $\Omega^{ai}_{\alpha q}$ has the form (3.28), eq. (3.26) is identically satisfied due to the properties of the γ–matrices in $d = 3$ (see Appendix **A**).

Thus in the case of the N=1, D=11 supermembrane the geometrodynamical condition (2.23) determines the minimal embedding of the world supersurface into the flat superspace.

In other words, the geometrodynamical condition (2.23) leads to the equations of motion of the twistor-like N=1, D=11 supermembrane, and, hence, as has been pointed out by Galperin and Sokatchev [36], when one introduces the geometrodynamical condition into a twistor–like supermembrane action with a Lagrange multiplier one may encounter

the problem with eliminating redundant propagating degrees of freedom of the Lagrange multiplier [36, 37, 38][1].

We conclude that in the framework of the geometrical approach the dynamics of the N=1, D=11 supermembrane is described by internal geometry on the world supersurface (i.e. d=2+1, n=8 supergravity) subject to the constraints (2.14), and by the superfields $\psi_{\{\alpha\beta\gamma\}\dot{q}}$ (3.28) satisfying the supersymmetric counterparts (2.38)–(2.40) of the Codazzi, Gauss and Ricci equation, which determine the minimal embedding of the world supersurface into the target superspace. With respect to the supergravity on the world supersurface $\psi_{\{\alpha\beta\gamma\}\dot{q}}$ can be regarded as the matter superfields.

[1]This point has been missed in [37]

Chapter 4

$D = 10$ superstrings

4.1 Type II superstrings

In the case of twistor–like type II superstrings in D=10 the situation is the same as in the N=1, D=11 supermembrane, i.e. the geometrodynamical condition (2.23) causes the strings to be on the mass shell and the embedding of a d=1+1, n=(8,8) worldsheet superspace into target superspace is minimal. The proof is almost straightforward for a type IIA superstring, since it can be obtained from the N=1, D=11 supermembrane by the dimensional reduction. For a D=10 IIB superstring, characterized by Grassmann Majorana–Weyl coordinates $\Theta^{1\underline{\mu}}$, $\Theta^{2\underline{\mu}}$ of the same chirality, solving for and getting the consequences of the twistor constraint (2.24) can be performed along the lines of ref. [36] for an N=2, D=3 twistor–like superstring, but using the Lorentz harmonics allows one to do this in a Lorentz covariant way.

Below we consider the consequences of the geometrodynamical condition and some features of the D=10 twistor–like IIA,B superstrings in the geometrical approach.

4.1.1 Lorentz harmonics in D=10

The spinor Lorentz harmonics which determine a local frame in a flat D=10 target superspace have the following form

$$v_{\underline{\mu}}^{\ \underline{\alpha}} = (v_{\underline{\mu}\ q}^{\ +}\ ,\ v_{\underline{\mu}\ \dot{q}}^{\ -}) \qquad \in \qquad Spin(1,9) \tag{4.1}$$

and the inverse harmonics are:

$$v_{\underline{\alpha}}^{\ \underline{\mu}} = \begin{pmatrix} v_q^{-\ \underline{\mu}} \\ v_{\dot{q}}^{+\ \underline{\mu}} \end{pmatrix} \qquad \in \qquad Spin(1,9) \tag{4.2}$$

$$v_{\underline{\mu}}^{\ \underline{\beta}} v_{\underline{\beta}}^{\ \underline{\nu}} = \delta_{\underline{\mu}}^{\ \underline{\nu}}, \qquad v_{\underline{\alpha}}^{\ \underline{\mu}} v_{\underline{\mu}}^{\ \underline{\beta}} = \begin{pmatrix} \delta_{qp} & 0 \\ 0 & \delta_{\dot{q}\dot{p}} \end{pmatrix} \tag{4.3}$$

($(+,-)$ stand for the spinor indices, while their pairs $(--,++)$ stand for the vector indices of SO(1,1) in a light–cone basis, and $\underline{\mu}, \underline{\nu} = 1, ..., 16$).

Because of the absence of the matrix of the charge conjugation of the Majorana–Weyl spinors in D=10 there is no direct linear expression of (4.2) in terms of (4.1) and vice versa.

The local vector frame $u^{\underline{a}}_{\underline{m}} = (u^{--}_{\underline{m}},\ u^{++}_{\underline{m}},\ u^{\ i}_{\underline{m}})$ can be composed either of (4.1)

$$\delta_{qp} u^{++}_{\underline{m}} = v^+_q \tilde{\Gamma}_{\underline{m}} v^+_p, \tag{4.4}$$

$$\delta_{\dot{q}\dot{p}} u^{--}_{\underline{m}} = v^-_{\dot{q}} \tilde{\Gamma}_{\underline{m}} v^-_{\dot{p}}, \tag{4.5}$$

$$\gamma^i_{q\dot{p}} u^i_{\underline{m}} = v^+_q \tilde{\Gamma}_{\underline{m}} v^-_{\dot{p}}, \tag{4.6}$$

or (4.2)

$$\delta_{qp} u^{--}_{\underline{m}} = v^-_q \Gamma_{\underline{m}} v^-_p, \tag{4.7}$$

$$\delta_{\dot{q}\dot{p}} u^{++}_{\underline{m}} = v^+_{\dot{q}} \Gamma_{\underline{m}} v^+_{\dot{p}}, \tag{4.8}$$

$$-\gamma^i_{q\dot{p}} u^i_{\underline{m}} = v^-_q \Gamma_{\underline{m}} v^+_{\dot{p}}. \tag{4.9}$$

The Lorentz harmonics $v_{\underline{\mu}}^{\ \underline{\alpha}}$ or $u^{\underline{a}}_{\underline{m}}$ parametrize a coset space $\frac{SO(1,9)}{SO(1,1)\times SO(8)}$. Note that if only half of the harmonics (for example, $v^{-\ \underline{\mu}}_q$) is involved in the description of a model, boost transformations of the form

$$\delta v^{-\ \underline{\mu}}_q = 0, \qquad \delta v^{+\ \underline{\mu}}_{\dot{q}} = b^{++i} \gamma^i_{q\dot{q}} v^{-\ \underline{\mu}}_q, \tag{4.10}$$

$$\delta u^{--}_{\underline{m}} = 0, \qquad \delta u^{++}_{\underline{m}} = b^{++i} u^i_{\underline{m}}, \qquad \delta u^i_{\underline{m}} = \frac{1}{2} b^{++i} u^{--}_{\underline{m}}, \tag{4.11}$$

become a symmetry of the model and can be used for reducing a number of independent variables in (4.2), (4.7)–(4.9) to that which parametrize an S^8 sphere being a compact subspace of $\frac{SO(1,9)}{SO(1,1)\times SO(8)}$. This is the case of a twistor–like formulation of an N=1 heterotic string [32], while a complete twistorization of the model [34] restores the coset space $\frac{SO(1,9)}{SO(1,1)\times SO(8)}$.

4.1.2 Geometrodynamical condition and twistor constraint for type II superstrings

One may consider n=(8,8) worldsheet superspace, where odd supervielbein components belong either to the same $(E^{+q},\ E^{-q})$ or the different $(E^{+q},\ E^{-\dot{q}})$ spinor representations of $SO(8)$. The former case is appropriate for the IIA superstring obtained by the dimensional reduction of the N=1, D=11 supermembrane [37], while for a IIB superstring we choose the latter case. In both cases we assume the worldsheet supergravity constraints (2.14) to imply

$$\{D_{-p}, D_{-q}\} = 2i\delta_{pq} D_{--}, \qquad \{D_{+p}, D_{+q}\} = 2i\delta_{pq} D_{++}, \qquad \{D_{-p}, D_{+q}\} = R_{-p+q}. \tag{4.12}$$

The embedding in question of the $n=(8,8)$ worldsheet superspace into the flat $D=10$

$$IIA \quad Z^M=(X^{\underline{m}},\Theta^{\underline{\mu}1},\Theta^2_{\underline{\mu}}), \qquad or \qquad IIB \quad Z^M=(X^{\underline{m}},\Theta^{\underline{\mu}1},\Theta^{\underline{\mu}2}) \tag{4.13}$$

target superspace is specified by the geometrodynamical condition

$$IIA: \qquad \Pi^{\underline{m}}_{\pm q}=D_{\pm q}X^{\underline{m}}-iD_{\pm q}\Theta^1\Gamma^{\underline{m}}\Theta^1-iD_{\pm q}\Theta^2\tilde{\Gamma}^{\underline{m}}\Theta^2=0, \tag{4.14}$$

or

$$IIB: \qquad \begin{cases}\Pi^{\underline{m}}_{+q}=D_{+q}X^{\underline{m}}-iD_{+q}\Theta^1\Gamma^{\underline{m}}\Theta^1-iD_{+q}\Theta^2\Gamma^{\underline{m}}\Theta^2=0\\ \Pi^{\underline{m}}_{-\dot{q}}=D_{-\dot{q}}X^{\underline{m}}-iD_{-\dot{q}}\Theta^1\Gamma^{\underline{m}}\Theta^1-iD_{-\dot{q}}\Theta^2\Gamma^{\underline{m}}\Theta^2=0\end{cases}. \tag{4.15}$$

The twistor constraints, which follow from (4.14), (4.15) and (4.12), are

$$IIA: \qquad \delta_{qp}(\gamma^{++}_{\alpha\beta}\Pi^{\underline{m}}_{++}+\gamma^{--}_{\alpha\beta}\Pi^{\underline{m}}_{--})=D_{\alpha q}\Theta^1\Gamma^{\underline{m}}D_{\beta p}\Theta^1+D_{\alpha q}\Theta^2\tilde{\Gamma}^{\underline{m}}D_{\beta p}\Theta^2, \tag{4.16}$$

$$IIB: \qquad \begin{cases}\delta_{qp}\Pi^{\underline{m}}_{++}=D_{+q}\Theta^1\Gamma^{\underline{m}}D_{+p}\Theta^1+D_{+q}\Theta^2\Gamma^{\underline{m}}D_{+p}\Theta^2,\\ \delta_{\dot{q}\dot{p}}\Pi^{\underline{m}}_{--}=D_{-\dot{q}}\Theta^1\Gamma^{\underline{m}}D_{-\dot{p}}\Theta^1+D_{-\dot{q}}\Theta^2\Gamma^{\underline{m}}D_{-\dot{p}}\Theta^2\\ D_{-\dot{q}}\Theta^1\Gamma^{\underline{m}}D_{+p}\Theta^1+D_{-\dot{q}}\Theta^2\Gamma^{\underline{m}}D_{+p}\Theta^2=0\end{cases} \tag{4.17}$$

($\gamma^{++}_{\alpha\beta}\equiv\delta^+_\alpha\delta^+_\beta, \quad \gamma^{--}_{\alpha\beta}\equiv\delta^-_\alpha\delta^-_\beta$).

By performing the dimensional reduction of the supermembrane relation (3.14), or by direct computation (Appendix B) one gets the general solution to the type IIA superstring twistor constraints (4.16) in the form

$$\begin{aligned}IIA: \qquad & D_{+q}\Theta^{\underline{\mu}1}=v^{\underline{\mu}}_{+q}, \qquad D_{+q}\Theta_{\underline{\mu}2}=0,\\ & D_{-q}\Theta^{\underline{\mu}1}=0, \qquad D_{-q}\Theta^2_{\underline{\mu}}=v_{-q\underline{\mu}},\end{aligned} \tag{4.18}$$

and to the type IIB twistor constraints (4.17) (see [36] for the N=2, D=3 superstring) in the form

$$\begin{aligned}IIB: \qquad & D_{+q}\Theta^{\underline{\mu}2}=-\tan\phi D_{+q}\Theta^{\underline{\mu}1}=-\sin\phi v^{\underline{\mu}}_{+q},\\ & D_{-\dot{q}}\Theta^{\underline{\mu}1}=\tan\phi D_{-\dot{q}}\Theta^{\underline{\mu}2}=\sin\phi v^{\underline{\mu}}_{-\dot{q}}.\end{aligned} \tag{4.19}$$

For both cases

$$\Pi^{\underline{m}}_{++}=u^{\underline{m}}_{--}, \qquad \Pi^{\underline{m}}_{--}=u^{\underline{m}}_{--}, \tag{4.20}$$

and the Virasoro conditions

$$(\Pi^{\underline{m}}_{++})^2=0=(\Pi^{\underline{m}}_{--})^2 \tag{4.21}$$

are identically satisfied. In (4.18)–(4.20) it is implied that the scale factor W is gauged away by the super–Weyl symmetry (2.19), and ϕ is a superfield parameter of the $SO(2)$ rotations which mix $\Theta^{\underline{\mu}1}$ and $\Theta^{\underline{\mu}2}$. The presence of this parameter distinguishes the IIB case from the IIA one where such mixing is impossible because of the different chirality of Θ^1 and Θ^2. Further analysis shows that ϕ is to be a constant.

Indeed, using (4.12) we derive the selfconsistency conditions for (4.19)

$$-2i\delta_{qp}(D_{++}\Theta^{\underline{\mu}2}+\tan\phi D_{++}\Theta^{\underline{\mu}1})=-D_{+\{p}\tan\phi D_{+q\}}\Theta^{\underline{\mu}1} \tag{4.22}$$

$$-2i\delta_{\dot{q}\dot{p}}(D_{--}\Theta^{\underline{\mu}1}+\tan\phi D_{--}\Theta^{\underline{\mu}2})=-D_{-\{\dot{p}}\tan\phi D_{\dot{q}\}}\Theta^{\underline{\mu}2}. \tag{4.23}$$

Contracting (4.22) with $v_{-r\underline{\mu}}$ and taking into account Eq. (4.19) we get

$$\cos\phi\delta_{r\{p}D_{+q\}}\tan\phi=-2i\delta_{\dot{q}\dot{p}}(D_{--}\Theta^{\underline{\mu}1}+\tan\phi D_{--}\Theta^{\underline{\mu}2})v_{-r\underline{\mu}}.$$

From which it follows that for any $r=p\neq q$

$$D_{+q}\tan\phi=0, \qquad \rightarrow \qquad D_{++}\tan\phi=0 \tag{4.24}$$

Following the same reasoning from (4.23) we get

$$D_{-\dot{q}}\tan\phi=0, \qquad \rightarrow \qquad D_{--}\tan\phi=0. \tag{4.25}$$

Thus,

$$\phi=\ const\ , \tag{4.26}$$

and one may choose new Grassmann coordinates $\hat{\Theta}^{1,2}$ as a linear combination of the old ones in such a way that the new variables satisfy the chirality conditions analogous to (4.18)

$$\begin{aligned} IIB: \qquad & D_{-\dot{q}}\hat{\Theta}^{\underline{\mu}1}\equiv D_{-\dot{q}}(\cos\phi\Theta^{\underline{\mu}1}-\sin\phi\Theta^{\underline{\mu}2})=0,\\ & D_{+q}\hat{\Theta}^{\underline{\mu}2}\equiv D_{+q}(\cos\Theta^{\underline{\mu}2}+\sin\phi\Theta^{\underline{\mu}1})=0. \end{aligned} \tag{4.27}$$

Note that the SO(2) rotations of $\Theta^{1,2}$ are not a symmetry of the IIB superstring [6, 47].

From (4.18), (4.27) it follows that

$$D_{--}\Theta^{\underline{\mu}1}=0=D_{--}\hat{\Theta}^{\underline{\mu}1}, \qquad D_{++}\Theta^{2}_{\underline{\mu}}=0=D_{++}\hat{\Theta}^{\underline{\mu}2}, \tag{4.28}$$

which are evidently dynamical equations. Hence, the geometrodynamical condition (4.16) leads to equations of motion of the $D=10$ type II superstrings (see [36]) and taking into account the results of Section **2.3**, we conclude that the embedding of the worldsheet n=(8,8) superspaces into the flat $D=10,\ N=2$ superspaces is minimal.

In the next subsection we will present the set of variables describing the dynamics of the $D=10$ IIA superstring in the geometrical approach.

4.1.3 Geometrical description of the $D=10$, IIA superstring

As we have already noted the most direct and simplest way to analyze the particular features of the D=10 IIA superstring in the geometrical approach is to perform dimensional reduction of the D=11 supermembrane equations from Chapter 3, and we only declare the results.

i) superfield W can be eliminated either with the use of the super–Weyl symmetry, or as a result of equations of motion;

ii) the induced $SO(1,1) \times SO(8)$ connection completely coincides with the intrinsic one (compare with (3.13)), i.e. $\Omega^{++--}(D) = 0, \quad \Omega^{ij}(D) = 0$, so that the internal properties of the worldsheet superspace are described by d=1+1, n=(8,8) supergravity subject to the constraints (4.12);

iii) the superstring modes transversal to the worldsheet are described by the pullback of the $\frac{SO(1,9)}{SO(1,1)\times SO(8)}$ Cartan form

$$
\begin{aligned}
\Omega^{++i} &= E^{-q} 2i\gamma^i_{q\dot{q}} \Psi^{+++}_{\dot{q}} + E^{--}\Omega_{--}^{\ ++\ i}, \\
\Omega^{--i} &= E^{+q} 2i\gamma^i_{q\dot{q}} \Psi^{---}_{\dot{q}} + E^{++}\Omega_{++}^{\ --\ i},
\end{aligned} \tag{4.29}
$$

where $\Psi^{+++}_{\dot{q}}, \ \Psi^{---}_{\dot{q}}$ are (anti)chiral Grassmann superfields

$$
\begin{aligned}
D_{+p}\Psi^{+++}_{\dot{q}} &= 0, \qquad \Rightarrow D_{++}\Psi^{+++}_{\dot{q}} = 0, \\
D_{-p}\Psi^{---}_{\dot{q}} &= 0, \qquad \Rightarrow D_{++}\Psi^{---}_{\dot{q}} = 0,
\end{aligned} \tag{4.30}
$$

subject to the Codazzi–like conditions (2.41) (compare with [64])

$$
\begin{aligned}
D_{-p}\Psi^{+++}_{\dot{q}} &= \tfrac{1}{2}\gamma^i_{p\dot{q}}\Omega_{--}^{\ ++\ i}, \\
D_{+p}\Psi^{---}_{\dot{q}} &= \tfrac{1}{2}\gamma^i_{p\dot{q}}\Omega_{++}^{\ --\ i},
\end{aligned} \tag{4.31}
$$

iv) the intrinsic $SO(1,1)\times SO(8)$ curvature tensor is expressed in terms of (4.29) through the doubly supersymmetric counterparts of the Gauss and Ricci equations (2.39), (2.40), (2.42), (2.43), for instance,

$$
\begin{aligned}
R^{++--}_{-p+q} &= 2\Psi^{+++}_{\dot{p}}\Psi^{---}_{\dot{q}}\gamma^i_{q\dot{q}}\gamma^i_{p\dot{p}}, \\
R^{ij}_{-p+q} &= 2\Psi^{+++}_{\dot{p}}\Psi^{---}_{\dot{q}}\gamma^{[i}_{q\dot{q}}\gamma^{j]}_{p\dot{p}},
\end{aligned} \tag{4.32}
$$

and we see that the spinor–spinor components of the curvature tensor are nilpotent, while the vector components are valuable (this situation should be understood yet);

v) upon getting the information about the geometrical objects mentioned in items i)–iv) one may restore (with taking into account appropriate boundary conditions) the coordinate functions $Z^{\underline{M}}(z^M)$ from the supercovariant forms

$$
\begin{aligned}
d\Theta^{\underline{\mu}1} &= E^{+q}v^{-\ \underline{\mu}}_{q} + E^{++}\Psi^{---}_{\dot{q}}v^{+\ \underline{\mu}}_{\dot{q}}, \\
d\Theta^{2}_{\underline{\mu}} &= E^{-q}v^{\ +}_{\underline{\mu}q} + E^{--}\Psi^{+++}_{\dot{q}}v^{\ -}_{\underline{\mu}\dot{q}}, \\
\Pi^{\underline{m}} &= E^{++}u^{\underline{m}}_{++} + E^{--}u^{\underline{m}}_{--}.
\end{aligned} \tag{4.33}
$$

4.2 D=10 twistor–like heterotic string

For comparison, let us make some comments on the N=1 supersymmetric part of the twistor–like heterotic string [29]–[34] in D=10.

It is well known that the geometrodynamical condition and the twistor constraint do not lead to the equations of motion of the heterotic string [29]–[34], so the embedding of heterotic (8,0) supersurface $(\xi^{\pm\pm}, \eta^{+q})$ into D=10, N=1 target superspace $(X^{\underline{m}}, \Theta^{\underline{\mu}})$ specified solely by the geometrodynamical condition is non–minimal.

The geometrodynamical condition and the twistor constraint are obtained from (4.14)–(4.17) by keeping only (+) SO(1,1) indices and have the following form, respectively:

$$\Pi^{\underline{m}}_{+q} \equiv D_{+q}X^{\underline{m}} - iD_{+q}\Theta\Gamma^{\underline{m}}\Theta = 0, \tag{4.34}$$

$$\delta_{qp}\Pi^{\underline{m}}_{++} = D_{+q}\Theta\Gamma^{\underline{m}}D_{+p}\Theta. \tag{4.35}$$

A consequence of (4.35) is one of the Virasoro conditions (4.21)

$$\Pi^{\underline{m}}_{++}\Pi_{++\ \underline{m}} = 0. \tag{4.36}$$

In the twistor-like formulation of refs. [29]–[33], as in the conventional Green–Schwarz formulation, the second Virasoro condition follows from varying an action with respect to the vielbeins, and hence *a priori* is not related to another twistor constraint.

If we work within this version then only half of the Lorentz spinor harmonics (4.1), (4.2) are involved, since upon performing an appropriate gauge fixing (see previous sections and Appendix B) one gets from (4.35) that

$$D_{+q}\Theta^{\underline{\mu}} = v^{\underline{\mu}}_{+q}, \tag{4.37}$$

$$\Pi^{\underline{m}}_{++} = u^{\underline{m}}_{++} = \frac{1}{8}v_{+q}\Gamma^{\underline{m}}v_{+q}, \tag{4.38}$$

but now we do not have any restrictions on $D_{--}\Theta^{\underline{\mu}}$, and $v^{\underline{\mu}}_{-\dot{q}}$ never appears in this version [32]. Hence, as we have noted in Subsection 4.1.1, such a model is invariant under the eight–parameter boost symmetry (4.10), (4.11) which allows one to reduce a number of independent variables contained in $v^{\underline{\mu}}_{+q}$ to that parametrizing an S^8 sphere [32]. This symmetry is broken by requiring

$$\Pi^{\underline{m}}_{--}u^{\ i}_{\underline{m}} = 0, \tag{4.39}$$

which, in assumption that $\Pi_{\underline{m}--}\Pi^{\underline{m}}_{--} = 0$ and $\Pi_{--}\Pi_{++} \neq 0$, implies (see eqs. (4.7)–(4.9))

$$\delta_{\dot{p}\dot{q}}\Pi^{\underline{m}}_{--} = \delta_{\dot{q}\dot{p}}u^{\underline{m}}_{--} = v_{-\dot{q}}\tilde{\Gamma}^{\underline{m}}v_{-\dot{p}}. \tag{4.40}$$

As a result all the spinor Lorentz harmonics become involved in to the game, and the relevant coset space is to be $\frac{SO(1,9)}{SO(1,1)\times SO(8)}$.

If one requires eq. (4.40) to be obtained from a heterotic string action functional, than one gets a completely twistorized heterotic string formulation considered in [34]. And it is just this version which is more appropriate for developing the geometrical approach in the framework discussed herein.

Thus, the embedding of the heterotic worldsheet into the flat target superspace is described by the following pullback of the supercovariant forms $d\Theta^{\underline{\mu}}$, $\Pi^{\underline{m}}$

$$d\Theta^{\underline{\mu}} = E^{+q} v_q^{-\underline{\mu}} + E^{++} D_{++}\Theta^{\underline{\mu}} + E^{--} D_{--}\Theta^{\underline{\mu}}, \tag{4.41}$$

$$\Pi^{\underline{m}} = E^{++} u^{\underline{m}}_{++} + E^{--} u^{\underline{m}}_{--} \tag{4.42}$$

and by the Maurer–Cartan equations (2.38)–(2.40) for $\Omega^{\underline{ab}}(D)$ constructed from the Lorentz harmonics. As in the case of the supermembrane and the type II superstrings, the selfconsistency conditions for (4.41), (4.42)

$$d\Pi^{\underline{m}} = -i d\Theta\Gamma^{\underline{m}} d\Theta, \tag{4.43}$$

$$dd\Theta = 0, \tag{4.44}$$

may further restrict the form of $\Omega^{\underline{ab}}(D)$, but one may convince oneself that without requiring for $\Theta^{\underline{\mu}}$ and $X^{\underline{m}}$ to satisfy additional equations obtained from the twistor–like heterotic string action [32] or [34], the embedding is non-minimal ($\Omega^{ai}_a \neq 0$). The detailed consideration of the heterotic case is beyond the scope of the present article.

Conclusion

We have performed a generalization of the geometrical approach to describing extended objects for studying the doubly supersymmetric twistor–like formulation of super–p–branes. Some basic features of embedding world supersurface into target superspace specified by the geometrodynamical condition (2.23) have been considered. It has been shown that the main attributes of the geometrical approach, such as the second fundamental form and extrinsic torsion of the embedded surface, and the Codazzi, Gauss and Ricci equations, have their doubly supersymmetric counterparts. At the same time the embedding of supersurface into target superspace has its particular features. For instance, in general, intrinsic and induced geometry on the supersurface may not directly coincide (for a chosen set of intrinsic geometry constraints), though they are related to each other by means of the geometrical equations, and the embedding may cause more rigid restrictions on the geometrical properties of the supersurface. This has been demonstrated with the examples of the N=1 twistor–like supermembrane in D=11 and the type II superstrings in D=10, where the geometrodynamical condition caused the embedded supersurface to be minimal and puts the theories on the mass shell. This feature seems to be related to the general problem of constructing off–shell superfield actions for models with the number of supersymmetries exceeding some "critical" value. In the cases considered world supersurface possesses n=(8,8) local supersymmetry which is indirectly related to an N=4

supergravity model in D=4 by dimensional reduction. And it is known that D=4, N=4 supergravity constraints put the theory on the mass shell. In the case of the twistor–like heterotic string (Section 4.2), where there are twice less supersymmetries on the world supersurface, the off–shell superfield formulations do exist [29]–[34].

Preliminary studies of N=2 twistor–like superparticles and N=2 superstrings in D=4 (with n=(2,2) worldsheet supersymmetry) in a version close to that of refs. [31] also show that the geometrodynamical condition does not result in equations of motion, and one may hope to write down a superfield action without facing the problem of propagating undesirable degrees of freedom.

As it was mentioned in the Introduction in the present paper we have not discussed the role of the Wess–Zumino term [6, 47, 29, 32]. The place of the Wess-Zumino form in the geometrical approach is to be understood yet, and we shall only make one comment. As we have seen, in the geometrical approach a basic role is played by the Maurer–Cartan equations for the one–forms determining the Lorentz group $SO(1, D-1)$ (eqs. (1.10)–(1.12)) and for the supercovariant one–forms on the target superspace (eqs. (2.2), (2.29), (2.30)). But in multidimensional curved target superspace supergravity is also characterized by a Grassmann antisymmetric, so called Kolb–Ramond, superfield. And it is just this superfield and its curl that contribute to the components of the Wess–Zumino form. The Kolb–Ramond superfield acquires geometrical meaning in a generalized group–manifold approach originated from a D=11 supergravity paper [71] [1], where generalized Maurer–Cartan equations for higher–degree differential forms (such as the Kolb–Ramond superfield) were proposed. Taking into consideration these generalized Maurer–Cartan equations together with eqs. (2.29), (2.30) should involve the Wess-Zumino form into the geometrical approach.

Beside the main purpose of the paper concerning the geometrical approach, we have also tried to demonstrate that the twistor–like spinors and the spinor Lorentz harmonics are closely related to each other and both describe the components of the local frame in target superspace. In this respect the Lorentz–harmonic formulation of super–p–branes developed in [39]–[44] can be regarded as a component version of the superfield twistor–like approach. The former is based on an action analogous to eq. (1.4) (with $u^a_{\underline{m}}$ being composed of harmonic (or twistor) components) and seems to be related to the geometrical approach in the most direct way. Thus if one tries to find some dynamical ground for developing the doubly supersymmetric geometrical approach to super–p–branes, it seems reasonable to construct a superfield generalization of the action (1.4) or (1.5). An example of such an action for N=1 massless superparticles in D=3,4 and 6 has been considered in [73].

Acknowledgements

[1] see also [72] for the case relevant to superstrings and supermembranes

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Appendix A: NOTATION and CONVENTION

$D = 11$ Γ–matrices, and spinor moving frame attached to supermembrane world supersurface

For describing the $D = 11$ supermembrane we use the following $SO(1,2) \times SO(8)$ invariant splitting of the charge conjugation matrix and the Γ-matrices in $D = 11$.

$$
\begin{aligned}
C^{\underline{\alpha}\underline{\beta}} = -C^{\underline{\beta}\underline{\alpha}} = &\quad diag\left(\epsilon^{\alpha\beta}\delta_{qp}, \ -\epsilon_{\alpha\beta}\delta_{\dot{q}\dot{p}}\right), \\
C^{-1}_{\underline{\alpha}\underline{\beta}} = &\quad diag\left(\epsilon_{\alpha\beta}\delta_{qp}, \ -\epsilon^{\alpha\beta}\delta_{\dot{q}\dot{p}}\right), \\
(\Gamma^{\underline{a}})_{\underline{\alpha}}{}^{\underline{\beta}} \equiv (\Gamma^{a}, \Gamma^{i}), & \\
(\Gamma^{a})_{\underline{\alpha}}{}^{\underline{\beta}} \equiv \left(\Gamma^{\underline{0}}, \ \Gamma^{\underline{9}}, \ \Gamma^{\underline{10}}\right) \equiv \left(\Gamma^{0}, \Gamma^{1}, \Gamma^{2}\right) = &\quad diag\left(\gamma^{a}{}_{\alpha}{}^{\beta}\delta_{qp}, -\gamma^{a}_{\beta}{}^{\alpha}\delta_{\dot{q}\dot{p}}\right), \\
(\Gamma^{i})_{\underline{\alpha}}{}^{\underline{\beta}} \equiv \left(\Gamma^{1}, \ldots, \Gamma^{8}\right) = &\quad \begin{bmatrix} 0 & \epsilon_{\alpha\beta}\gamma^{i}_{q\dot{p}} \\ -\epsilon^{\alpha\beta}\tilde{\gamma}^{i}_{\dot{q}p} & \end{bmatrix}
\end{aligned}
\tag{45}
$$

where $\underline{\alpha} = \begin{pmatrix} {}^{\alpha}_{q}, & {}_{\alpha\dot{q}} \end{pmatrix}$ is composed of $SO(1,2) \times SO(8)$ spinor indices, $\underline{\mu}, \underline{\alpha} = 1, \ \ldots, \ 32$ are the spinor indices of $SO(1,10)$; $\alpha, \beta = \ 1, \ 2$ are the spinor indices of $SO(1,2)$, and $q, \ p = 1, \ldots, 8$; $\dot{q}, \ \dot{p} = 1$, $\ldots$, 8 are s- and c- spinor indices of SO(8), respectively; $\tilde{\gamma}^{i}_{\dot{q}p} \equiv \gamma^{i}_{p\dot{q}}$ are $d = 8$ γ-matrices, $\gamma^{a}{}_{\beta}{}^{\alpha}$ are $d = 3$ γ–matrices, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ ($\epsilon^{12} = \epsilon_{12} = i$) is the $d = 3$ charge conjugation matrix.

Note that $D = 11$ as well as $d = 3$ gamma matrices with both indices being upper or lower are symmetric:

$$
\begin{aligned}
\left(\Gamma^{\underline{m}}C^{-1}\right)^{T} &= \left(\Gamma^{\underline{m}}C^{-1}\right), \\
(\epsilon\gamma^{a})^{T} &= (\epsilon\gamma^{a}).
\end{aligned}
\tag{46}
$$

In the main text and below we skip C and ϵ in the formulas like (46) and write, for example $(\Gamma^{\underline{m}}C^{-1})_{\underline{\alpha}\underline{\beta}} \equiv (\Gamma^{\underline{m}})_{\underline{\alpha}\underline{\beta}}$.

The Lorentz harmonics form a 32×32 matrix $v^{\underline{\alpha}}_{\underline{\mu}}$ of $Spin(1,10)$:

$$
v_{\underline{\mu}}{}^{\underline{\alpha}} = \left(v_{\underline{\mu}q}{}^{\alpha}, \ v_{\underline{\mu}\ \alpha\dot{q}}\right).
\tag{47}
$$

We use the "left" action of the charge conjugation matrices for rising and lowering spinor indices in D=11 and d=3:

$$
v^{\underline{\mu}}{}_{\beta q} \equiv C^{\underline{\mu}\underline{\nu}} v_{\underline{\nu}\beta q}, \qquad v_{\underline{\mu}\ \beta q} \equiv C^{-1}_{\underline{\mu}\underline{\nu}} v^{\underline{\nu}}{}_{\beta q}, \qquad v^{\underline{\mu}}{}_{q}{}^{\alpha} \equiv \epsilon^{\alpha\beta} v^{\underline{\nu}}{}_{\beta q}, \qquad v^{\underline{\mu}}_{\alpha q} \equiv \epsilon_{\alpha\beta} v^{\underline{\nu}}{}_{q}{}^{\beta}.
\tag{48}
$$

The requirement that the matrix (47) takes its values in the group $Spin(1,10)$ (which is the double-covering group of $SO(1,10)$) can be ensured by imposing the following "harmonic" conditions

$$\Xi \equiv v_{\underline{\mu}}^{\underline{\alpha}} C^{\underline{\mu}\underline{\nu}} v_{\underline{\nu}}^{\underline{\beta}} - C^{\underline{\alpha}\underline{\beta}} = 0, \tag{49}$$

$$\Xi_{\underline{m}_1\underline{m}_2}^{\underline{a}} \equiv v_{\underline{\mu}}^{\underline{\alpha}} (\Gamma_{\underline{m}_1\underline{m}_2})^{\underline{\mu}\underline{\nu}} v_{\underline{\nu}}^{\underline{\beta}} (\Gamma^{\underline{a}})_{\underline{\alpha}\underline{\beta}} = 0, \tag{50}$$

$$\Xi_{\underline{m}_1\cdots\underline{m}_5}^{\underline{a}} \equiv v_{\underline{\mu}}^{\underline{\alpha}} \left(\Gamma_{\underline{m}_1\cdots\underline{m}_5}\right)^{\underline{\mu}\underline{\nu}} v_{\underline{\nu}}^{\underline{\beta}} (\Gamma^{\underline{a}})_{\underline{\alpha}\underline{\beta}} = 0. \tag{51}$$

Eqs. (49), (48) allow one to determine the matrix inverse to $v_{\underline{\mu}}^{\underline{\alpha}}$ by use of the same variables $v_{\underline{\mu}q}^{\ \alpha}, v_{\underline{\mu}\ \alpha\dot{q}}$:

$$v_{\underline{\beta}}^{\underline{\nu}} \equiv (v^{-1})_{\underline{\beta}}^{\underline{\nu}} = \left(-v_{\beta q}^{\underline{\nu}}, v_{\ \dot{q}}^{\underline{\nu}\beta}\right). \tag{52}$$

One can see that not all the relations encoded in (49)–(51) are independent (they "kill" $969 = 1024\text{-}55$ degrees of freedom [42]). An independent subset can be chosen in different ways. For instance, in (49) one can take the only independent condition to be that the harmonics have unit norm. Then (50), (51) contain just the same information as (2.7), or (2.10)–(2.12).

Hence, among the 1024 components of $v_{\underline{\mu}q}^{\alpha}, v_{\underline{\mu}\ \alpha\dot{q}}$ only $55 = dim\ SO(1,10)$ are independent. Among the latter $31 = 3 + 28 = dim\ SO(1,2) + dim\ SO(8)$ can be gauged away by $SO(1,2)\times SO(8)$ local symmetry of the supermembrane theory. Thus, $v_{\underline{\mu}q}^{\alpha}, v_{\underline{\mu}\ \alpha\dot{q}}$ parametrize a coset space $\frac{SO(1,10)}{SO(1,2)\times SO(8)}$.

$D = 10$ Γ–matrices, and spinor moving frame attached to superstring worldsheet

For $D = 10$ superstrings the vector indices take ten values $\underline{m}, \underline{a} = 0, 1, \ldots, 9$ and the dimension of the Majorana–Weyl spinor representation is 16: $\underline{\alpha}, \underline{\mu} = 1, \ldots, 16$.

For making computations we use the following $SO(1,1)\times SO(8)$ invariant realization of 16×16 Γ-matrices:

$$\Gamma_{\underline{\alpha}\underline{\beta}}^{\underline{0}} = \ diag(\delta_{qp}, \delta_{\dot{q}\dot{p}}) = \tilde{\Gamma}^{\underline{0}\ \underline{\alpha}\underline{\beta}}, \tag{53}$$

$$\Gamma_{\underline{\alpha}\underline{\beta}}^{\underline{9}} = \ diag(\delta_{qp}, -\delta_{\dot{q}\dot{p}}) = -\tilde{\Gamma}^{\underline{9}\ \underline{\alpha}\underline{\beta}}, \tag{54}$$

$$\Gamma_{\underline{\alpha}\underline{\beta}}^{i} = \begin{pmatrix} 0 & \gamma_{q\dot{p}}^{i} \\ \tilde{\gamma}_{\dot{q}p}^{i} & 0 \end{pmatrix} = -\tilde{\Gamma}^{i\ \underline{\alpha}\underline{\beta}}, \tag{55}$$

$$\Gamma_{\underline{\alpha}\underline{\beta}}^{++} \equiv (\Gamma^{\underline{0}} + \Gamma^{\underline{9}})_{\underline{\alpha}\underline{\beta}} = \ diag(\ 2\delta_{qp},\ 0) = -(\tilde{\Gamma}^{\underline{0}} - \tilde{\Gamma}^{\underline{9}})^{\underline{\alpha}\underline{\beta}} = \tilde{\Gamma}^{--\ \underline{\alpha}\underline{\beta}}, \tag{56}$$

$$\Gamma_{\underline{\alpha}\underline{\beta}}^{--} \equiv (\Gamma^{\underline{0}} - \Gamma^{\underline{9}})_{\underline{\alpha}\underline{\beta}} = \ diag(\ 0,\ 2\delta_{\dot{q}\dot{p}}) = (\tilde{\Gamma}^{\underline{0}} + \tilde{\Gamma}^{\underline{9}})^{\underline{\alpha}\underline{\beta}} = \tilde{\Gamma}^{++\ \underline{\alpha}\underline{\beta}}. \tag{57}$$

Note that with respect to their properties the matrices $\Gamma,\ \tilde{\Gamma}$ are closer to the D=4 Pauli matrices rather then to the Dirac matrices.

Let us also stress the absence of the charge conjugation matrix in the $D = 10$ Majorana–Weyl spinor representation, so the Γ–matrices always have both spinor indices down and $\tilde{\Gamma}$ have both spinor indices up. As a result there is no linear expression for the inverse Lorentz harmonics $(v^{\underline{\mu}}_{+q},\ v^{\underline{\mu}}_{-\dot{q}})$ through $(v^{+}_{\underline{\mu}q},\ v^{-}_{\underline{\mu}\dot{q}})$. They are related by the requirement that the spinor harmonics and their inverse define one and the same composed vector frame:

$$\begin{aligned} u^{++}_{\underline{m}} = & \quad \tfrac{1}{8} v^{+}_{q} \tilde{\sigma}_{\underline{m}} v^{+}_{q} = \tfrac{1}{8} v^{+}_{\dot{q}} \sigma_{\underline{m}} v^{+}_{\dot{q}} \\ u^{--}_{\underline{m}} = & \quad \tfrac{1}{8} v^{-}_{\dot{q}} \tilde{\sigma}_{\underline{m}} v^{-}_{\dot{q}} = \tfrac{1}{8} v^{-}_{q} \sigma_{\underline{m}} v^{-}_{q} \\ u^{i}_{\underline{m}} = & \quad \tfrac{1}{8} \gamma^{i}_{q\dot{q}} v^{+}_{q} \tilde{\sigma}_{\underline{m}} v^{-}_{\dot{q}} = -\tfrac{1}{8} \gamma^{i}_{q\dot{q}} v^{-}_{q} \sigma_{\underline{m}} v^{+}_{\dot{q}} . \end{aligned} \tag{58}$$

The irreducible harmonic conditions for the matrix $(v^{+}_{\underline{\mu}q},\ v^{-}_{\underline{\mu}\dot{q}})$ to take values in $Spin(1,9)$ are:

$$u^{\underline{m}}_{\underline{a}} \quad \Xi^{\underline{a}}_{\underline{m}_1 \cdots \underline{m}_4 \underline{m}} = u^{\underline{m}}_{\underline{a}} Sp(v^{T} \tilde{\Gamma}_{\underline{m} \cdots \underline{m}_4} v \Gamma^{\underline{a}}) = 0, \tag{59}$$

$$\Xi_0 \equiv u^{--}_{\underline{m}} u^{\underline{m}++} - 2 = 0, \tag{60}$$

(where eqs. (58) for $u^{\pm\pm}_{\underline{m}}$ are implied). Then

$$\Xi^{\underline{a}}_{\underline{m}_1 \cdots \underline{m}_5} = 0$$

is identically satisfied.

For the detailed discussion of the Lorentz harmonics in D=10 see refs. [62, 49, 50, 63, 40, 41].

Appendix B

Here, for the $D = 11$, N=1 supermembrane and $D = 10$, IIA superstring, we present a direct proof that eqs. (2.25), (2.26) expressing $D_{\alpha q} \Theta^{\underline{\mu}}$ and $\Pi^{\underline{m}}_{a}$ in terms of the Lorentz harmonics are the consequence of the twistor constraint (2.24):

$$\delta_{qp} \gamma^{a}{}_{\alpha\beta} \Pi_{a}{}^{\underline{m}} = D_{\alpha q} \Theta \Gamma^{\underline{m}} D_{\beta p} \Theta. \tag{61}$$

Solving the twistor constraint for supermembrane in $D = 11$.

Let us choose in target superspace a local moving frame (2.10)–(2.12):

$$\delta_{\hat{q}\hat{p}} (\gamma_{\hat{a}})_{\hat{\alpha}\hat{\beta}} u^{\hat{a}}_{\underline{m}} = v_{\hat{\alpha}\hat{q}} \Gamma_{\underline{m}} v_{\hat{\beta}\hat{p}}, \tag{62}$$

$$\delta_{\dot{\hat{q}}\dot{\hat{p}}} (\gamma_{\hat{a}})^{\hat{\alpha}\hat{\beta}} u^{\hat{a}}_{\underline{m}} = v^{\hat{\alpha}}_{\dot{\hat{q}}} \Gamma_{\underline{m}} v^{\hat{\beta}}_{\dot{\hat{p}}}, \tag{63}$$

$$\delta^{\hat{\alpha}}_{\hat{\beta}} \gamma^{\hat{i}}_{\hat{q}\dot{\hat{p}}} u^{\hat{i}}_{\underline{m}} = v_{\hat{\alpha}\hat{q}} \Gamma_{\underline{m}} v^{\hat{\beta}}_{\dot{\hat{p}}}. \tag{64}$$

Note that *a priori* the local $SO(1,2)\times SO(8)$ group acting from the right on the components of the local frame does not coincide with the local $SO(1,2)\times SO(8)$ group related to the world supersurface. This is indicated by hats on $SO(1,2)\times SO(8)$ indices.

By use of SO(1,D-1) transformations the local Lorentz frame can always be chosen in such a way that

$$\Pi_a^{\underline{m}} u_{\underline{m}}^{\hat{i}} = 0. \tag{65}$$

Then multiplying, respectively, the l.h.s. and the r.h.s. of (61) by the l.h.s. and the r.h.s. of (62)–(64) we get

$$\delta_{qp}\gamma^b_{\alpha\beta}F_b^{\hat{a}} = F_{\alpha q}^{\underline{\alpha}}(\Gamma^{\hat{a}})_{\underline{\alpha\beta}}F_{\beta p}^{\underline{\beta}}, \tag{66}$$

$$F_{\alpha q}^{\underline{\alpha}}(\Gamma^{\hat{i}})_{\underline{\alpha\beta}}F_{\beta p}^{\underline{\beta}} = 0, \tag{67}$$

where

$$F_b{}^{\hat{a}} \equiv \Pi_b^{\underline{m}} u_{\underline{m}}^{\hat{a}} \tag{68}$$

and

$$F_{\alpha q}^{\underline{\alpha}} \equiv D_{\alpha q}\Theta^{\underline{\mu}} v_{\underline{\mu}}^{\underline{\alpha}} \equiv (A_{\alpha q}{}^{\hat{\alpha}}{}_{\hat{p}}, B_{\alpha q\ \hat{\alpha}\dot{\hat{p}}}). \tag{69}$$

Let us begin with considering eq.(67). Using the explicit form of the Γ–matrices (45) and (69) we get

$$A_{\alpha q\hat{p}}^{\hat{\alpha}}\gamma^{\hat{i}}_{\hat{p}\dot{\hat{p}}}B_{\beta p\hat{\alpha}\dot{\hat{p}}} + B_{\alpha q\hat{\alpha}\dot{\hat{p}}}\tilde{\gamma}^{\hat{i}}_{\dot{\hat{p}}\hat{p}}A_{\beta p\hat{p}}^{\hat{\alpha}} = 0. \tag{70}$$

Suppose that one of the matrices, for example A, is non–degenerate

$$det(A_{\alpha q}{}^{\hat{\alpha}}{}_{\hat{p}}) \neq 0 \tag{71}$$

(otherwise the solution we would got corresponded to a null super–p–brane (see [33] for the string case). Then we can rewrite (70) as follows:

$$(A^{-1}B)_{\hat{\alpha}\hat{p}\hat{\beta}\dot{\hat{q}}}\tilde{\gamma}^{\hat{i}}_{\dot{\hat{q}}\hat{q}} + (A^{-1}B)_{\hat{\beta}\hat{q}\hat{\alpha}\dot{\hat{q}}}\tilde{\gamma}^{\hat{i}}_{\dot{\hat{q}}\hat{p}} = 0. \tag{72}$$

Decomposing the matrix $(A^{-1}B)$ into the SO(1,2) irreducible parts

$$(A^{-1}B)_{\hat{\alpha}\hat{p}\hat{\beta}\dot{\hat{q}}} \equiv \epsilon_{\hat{\alpha}\hat{\beta}}G_{0\hat{p}\dot{\hat{p}}} + \gamma^{\hat{a}}_{\hat{\alpha}\hat{\beta}}G_{\hat{a}\hat{p}\dot{\hat{q}}}, \tag{73}$$

and substituting (73) into (72), we get two equations:

$$G_{0[\hat{p}\dot{\hat{q}}}\tilde{\gamma}^{i}_{\dot{\hat{q}}\hat{q}]} = 0, \tag{74}$$

and

$$G_{\hat{a}\{\hat{p}\dot{\hat{q}}}\tilde{\gamma}^{\hat{i}}_{\dot{\hat{q}}\hat{q}\}} = 0. \tag{75}$$

Decomposing G_0 and G_a onto the $SO(8)$ irreducible parts

$$G_{0\hat{p}\dot{\hat{q}}} = G_0^{\hat{j}}\tilde{\gamma}^{\hat{j}}_{\hat{p}\dot{\hat{q}}}, + G_0^{\hat{j}\hat{k}\hat{l}}\tilde{\gamma}^{\hat{j}\hat{k}\hat{l}}_{\hat{p}\dot{\hat{q}}}, \tag{76}$$

$$G_{\hat{a}\hat{p}\dot{\hat{q}}} = G^{\hat{j}}_{\hat{a}}\tilde{\gamma}^{\hat{j}}_{\hat{p}\dot{\hat{q}}}, + G^{\hat{j}\hat{k}\hat{l}}_{\hat{a}}\tilde{\gamma}^{\hat{j}\hat{k}\hat{l}}_{\hat{p}\dot{\hat{q}}}, \tag{77}$$

and substituting (76), (77) into eqs. (74), (75) we obtain

$$(G^{[\hat{j}}_0 \delta^{\hat{k}]\hat{i}} + G^{[\hat{j}\hat{k}\hat{i}]}_0)\tilde{\gamma}^{\hat{j}\hat{k}}_{\hat{p}\dot{\hat{q}}} = 0, \tag{78}$$

$$G^{\hat{i}}_{\hat{a}}\delta_{\hat{p}\dot{\hat{q}}} + G^{\hat{j}\hat{k}\hat{l}}_{\hat{a}}\gamma^{\hat{j}\hat{k}\hat{l}\hat{i}}_{\hat{p}\dot{\hat{q}}} = 0, \tag{79}$$

from which it follows that all the components of G_0 and $G_{\hat{a}}$ vanish, and hence

$$(A^{-1}B)_{\hat{\alpha}\hat{p}\hat{\beta}\dot{\hat{q}}} = 0 = B_{\alpha p\hat{\beta}\dot{\hat{q}}} \tag{80}$$

since the matrix A was supposed to be non–degenerate. Thus eq. (67) is valid if and only if

$$D_{\alpha p}\Theta^{\underline{\mu}} v_{\underline{\mu}\hat{\alpha}\dot{\hat{q}}} = 0$$

Let us turn to eq.(66). Substituting (69) and taking into account (80) we can rewrite eq.(66) in the form

$$A_{\alpha q}{}^{\hat{\alpha}}{}_{\hat{p}}(\gamma^{\hat{a}})_{\hat{\alpha}\hat{\beta}} A_{\beta p}{}^{\hat{\beta}}{}_{\hat{p}} = \delta_{qp}(\gamma^b)_{\alpha\beta} F^{\hat{a}}_b, \tag{81}$$

where $F^{\hat{a}}_b$ is defined by the eqs. (65), (68).

To analyse eq.(81) let us expend the matrix A in a complete basis of the space of 2×2 matrices:

$$A_{\alpha q}{}^{\hat{\beta}}{}_{\hat{p}} = a_{q\hat{p}}\delta^{\hat{\beta}}_{\alpha} + b^a_{q\hat{p}}(\gamma_a)_{\alpha}{}^{\hat{\beta}}. \tag{82}$$

Note that eq. (82) is invariant only under the diagonal subgroup $SO(1,2)$ of the $SO(1,2)\times SO(1,2)$.

With A being in the form (82), eq. (81) splits, in particular, into equations

$$(a(b^a)^T)_{p\hat{q}} = 0, \tag{83}$$

which, if the matrix a is nonsingular ($det\ a \neq 0$), result in

$$(b^a)_{p\hat{q}} = 0. \tag{84}$$

Then from (81) we get

$$(a\ a^T)_{qp}\eta^{ab} = \delta_{qp}F^{ab}. \tag{85}$$

An evident consequence of (85) is that F^{ab} is proportional to the unit matrix

$$F^{ab} = W^2\eta^{ab}, \tag{86}$$

which results in (see (2.26))

$$\Pi^{\underline{m}}_a = W^2 u^{\underline{m}}_a, \tag{87}$$

and hence

$$\Pi^{\underline{m}}_a \Pi_{\underline{m}b} = W^2\eta_{ab}.$$

On the other hand, eq. (85) means that the matrix a is proportional to an SO(8) matrix L:

$$a_{q\hat{p}} = W L_{q\hat{p}}, \tag{88}$$

$$(LL^T)_{qp} = \delta_{qp}. \tag{89}$$

Now one can use the SO(8) transformations of the local frame to fix $L_{q\hat{p}} = \delta_{q\hat{p}}$, thus remaining with only one SO(8) relevant to the supermembrane world supersurface. As the final result we get

$$D_{\alpha q}\Theta^{\underline{\mu}} = W v^{\underline{\mu}}_{\alpha q}, \qquad \Pi^{\underline{m}}_{a} = W^2 u^{\underline{m}}_{a}. \tag{90}$$

Solving the twistor constraint for type II superstrings in D=10.

For the type IIA superstring this can be done either by performing the dimensional reduction of the supermembrane relations (90) or by direct computation completely analogous to that for the supermembrane case. Thus we only present the result:

$$D_{+q}\Theta^{1\underline{\mu}} = W v^{\underline{\mu}}_{+q}, \qquad D_{-q}\Theta^{1\underline{\mu}} = 0; \tag{91}$$

$$D_{+q}\Theta^{2}_{\underline{\mu}} = 0, \qquad D_{-q}\Theta^{2}_{\underline{\mu}} = W v_{\underline{\mu}+q}; \tag{92}$$

and

$$\Pi^{\underline{m}}_{++} = W^2 u^{\underline{m}}_{++}, \qquad E^{\underline{m}}_{--} = W^2 u^{\underline{m}}_{--}. \tag{93}$$

The latter equations ensure the validity of the Virasoro conditions.

The (anti)chirality of Θ^1, Θ^2 (eqs. (91), (92) immediately follows from (90) if one takes into account that in D=10 $v^{\underline{\mu}}_{-q}$ and $v_{\underline{\mu}+q}$ are not present (see Appendix A).

To convince the reader that Eqs. (91)–(93) correspond to the general solution, below we indicate the main steps of the straightforward proof. As for the supermembrane, one can choose the harmonic variables in such a way, that

$$\Pi^{\underline{m}}_{a} u^{i}_{\underline{m}} = 0, \tag{94}$$

or

$$\Pi^{\underline{m}}_{a} = \underline{F}_{a\bar{b}} u^{\bar{b}}_{\underline{m}}, \tag{95}$$

with

$$\underline{F}_{a}{}^{\bar{b}} \equiv \Pi^{\underline{m}}_{a} u^{b}_{\underline{m}}, \tag{96}$$

then by use of Eqs. (58) the twistor constraints

$$D_{\alpha q}\Theta^{1\underline{\mu}}\Gamma^{\underline{m}}_{\underline{\mu\nu}} D_{\beta p}\Theta^{1\underline{\nu}} + D_{\alpha q}\Theta^{2}_{\underline{\mu}}\tilde{\Gamma}_{\underline{m}\underline{\mu\nu}} D_{\beta p}\Theta^{2}_{\underline{\nu}} = \delta_{qp}(\gamma^{++}_{\alpha\beta}\Pi^{\ \underline{m}}_{++} + \gamma^{--}_{\alpha\beta}\Pi^{\ \underline{m}}_{--}) \tag{97}$$

can be rewritten as follows:

$$F^{\hat{\underline{\alpha}}}_{\alpha q}(\Gamma^{\bar{a}})_{\underline{\hat{\alpha}\hat{\beta}}} F^{\hat{\underline{\beta}}}_{\beta p} = \delta_{qp}\gamma^{b}_{\alpha\beta}\underline{F}^{\bar{a}}_{b}, \tag{98}$$

$$F^{\hat{\underline{\alpha}}}_{\alpha q}(\Gamma^{i})_{\hat{\underline{\alpha}}\hat{\underline{\beta}}}F^{\hat{\underline{\beta}}}_{\beta p}=0, \tag{99}$$

where the matrix $F^{\hat{\underline{\alpha}}}_{\alpha q}$ is defined by the relation

$$F^{\hat{\underline{\alpha}}}_{\alpha q}=(D_{\alpha q}\Theta^{1\underline{\mu}}v^{\underline{\alpha}}_{\underline{\mu}},D_{\alpha q}\Theta^{2}_{\underline{\mu}}v^{\underline{\mu}}_{\underline{\alpha}})=(A_{\alpha q}{}^{\bar{\alpha}}{}_{A},B_{\alpha q}{}_{\bar{\alpha}\dot{A}}), \tag{100}$$

the indices $\hat{\underline{\alpha}}$, $\hat{\underline{\beta}}=1,\ldots,32$ and $(\Gamma^{\underline{a}})_{\hat{\underline{\alpha}}\hat{\underline{\beta}}}$ are ten of the eleven Γ–matrices (45). The transverse part (99) has the same form as eq. (67), thus, the problem under consideration is reduced to that having been solved for the supermembrane, and using the same reasoning we finally arrive at the solution (91)–(93).

The case of a twistor–like IIB superstring can be analyzed following the same group–theoretical reasoning as above (see [35] for $D=3$, $N=2$ superstring) with the result having been presented in Chapter 4.

Appendix C

In which we show that for the two sets of the Maurer–Cartan equations, namely (2.29), (2.30) and (2.38)–(2.40), their spinor–spinor components ensure the validity of the rest, if the number of world surface supersymmetries is more than one. This means that the irreps of the spinor–vector and vector–vector components either coincide with that of spinor–spinor ones or can be obtained by acting on the latter with the spinor covariant derivatives. Thus, for getting all the consequences of the Maurer–Cartan equations it is sufficient to consider just the spinor–spinor components. Of course, another choice of the independent equations is possible.

The situation is analogous to Bianchi identity theorems in super–Yang–Mills [65] and supergravity theories [66]–[68].

Let us consider eqs. (2.29), (2.30) and, for convenience, denote their l.h.s. by $M^{\underline{M}}=(M^{\underline{m}},M^{\underline{\mu}})$

$$M^{\underline{m}}\equiv\frac{1}{2}E^{B}E^{C}M^{\underline{m}}_{CB}\equiv d\Pi^{\underline{m}}+id\Theta\Gamma^{\underline{m}}d\Theta=0, \tag{101}$$

$$M^{\underline{\mu}}\equiv\frac{1}{2}E^{B}E^{C}M^{\underline{\mu}}_{CB}\equiv dd\Theta^{\underline{\mu}}=0. \tag{102}$$

Note that when Π^{m} and $\Theta^{\underline{\mu}}$ are written in terms of the harmonics (see (93)), equations (101), (102) become nontrivial.

The components of (101) and (102) in the basis of supercovariant two–forms

$$E^{\alpha q}E^{\beta p},\qquad E^{a}E^{\beta p},\qquad E^{a}E^{b}$$

are

$$M^{\underline{M}}_{\alpha q\beta p}=0, \tag{103}$$

$$M^{\underline{M}}_{\alpha qb}=0, \tag{104}$$

$$M^{\underline{M}}_{ab}=0. \tag{105}$$

The spinor–spinor components (eq. (103)) are just equations (2.22) and (2.24).

The integrability conditions for (101), (102), which are analogous to the Bianchi identities [65], [66, 67] read

$$I^{\underline{m}} \equiv dM^{\underline{m}} = 0, \tag{106}$$

$$I^{\underline{\mu}} \equiv dM^{\underline{\mu}} = 0. \tag{107}$$

Suppose that eqs. (103) hold, then, using the torsion constraints (2.14), we can derive from the spinor component equations $I^{\underline{M}}_{\alpha q\ \beta p\ \gamma r} = 0$ contained in (106), (107) the identity

$$\delta_{qp}\gamma^{a}{}_{\alpha\beta}M^{\underline{M}}_{a\ \gamma r} + \delta_{rq}\gamma^{a}{}_{\gamma\alpha}M^{\underline{M}}_{a\ \beta p} + \delta_{pr}\gamma^{a}{}_{\beta\gamma}M^{\underline{M}}_{a\ \alpha q} = 0 \tag{108}$$

For a world surface superspace with $n > 1$ we can put in eq.(108) $p = r \neq q$ for each value of q and get

$$\gamma^{a}{}_{\beta\gamma}M^{\underline{M}}_{\alpha q a} = 0 = M^{\underline{M}}_{\alpha q a}. \tag{109}$$

Hence, (104) is a consequence of (103). To show that (105) is also a consequence of (103) one should consider other components of (106), (107) with taking into account that $M^{\underline{M}}_{\alpha q\beta p}$ and $M^{\underline{M}}_{\alpha q b}$ vanish.

For the $SO(1, D-1)$ Maurer–Cartan equations (2.38)–(2.40) the proof can be performed in the same way.

In the case of n=1 world surface supersymmetry eqs. (104), (105) may produce independent consequences, as one may already see from eq. (108).

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